

Full Length Research Paper

Variational iteration algorithm-II for solving the system of third order non-linear integro-differential equations

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Variational iteration method is widely used for solving nonlinear, integro-differential equations. Through careful investigation of the iteration formulas of the earlier variational iteration algorithm (VIM), we find unnecessary repeated calculations in each iteration. To overcome this shortcoming, the variational iteration algorithm-II (He et al., 2010) will be used in this paper to solve the system of non linear integro-differential equations. Beside this, the comparison of the exact solution with approximated solution by VIM-II is illustrated by the graphs. Several examples are given to verify the reliability and efficiency of the method.

Key words: Variational iteration algorithm-II (VIM), system of nonlinear integro-differential equations.

INTRODUCTION

The higher-order integro differential equations arise in mathematical, applied and engineering sciences, astrophysics, solid state physics, astronomy, fluid dynamics, beam theory, fiber optics, glass-forming process and chemical reaction-diffusion models (Agarwal, 1983; Hashim and Comput, 2006; Morchalo, 1975; Kythe and Puri, 2002; Wang et al., 2007). Several techniques including Adomian decomposition, variational iteration and homotopy perturbation method have been used to investigate integro-differential (Hashim and Comput, 2006; El-Sayed et al., 2004; Saberi-Nadjafi and Tamamgar, 2008; Abbasbandy and Shivanian, 2009; Abbasbandy et al., 2008, 2009; Sweilam, 2007; Xu et al., 2009; Mohyud-Din and Naturforsch, 2010; Yildirim and Naturforsch, 2010; Wang and He, 2007; Hesameddini and Latifizadeh, 2009; Ghasemi et al., 2007; Yusufoglu, 2007; Yildirim, 2008; El-Shahed, 2005).

In a recent review article, He et al. (2010) summarized three variational iteration algorithms. Here, we will use the variational iteration algorithm-II for the study. It is an alternative approach to system of third order non-linear

integro-differential equations using the variational iteration method. The approach used is based on the variational iteration method (VIM) proposed by the Chinese researcher J. H. He. This method has wider application because; it reduces the size of computation and is a very powerful mathematical tool for various kinds of linear and nonlinear problems as well as system of ordinary differential equations, partial differential equations and integro-differential equations. In a recent publication, He et al. (2010) stated that "The Variational Iteration Method Which Should be Followed", the problem was completely eliminated, and a new iteration algorithm was suggested, and the algorithm was termed as the variational iteration algorithm-II. To the best of our knowledge, this paper represents the first application of variational iteration algorithm-II to solve the system of nonlinear integro-differential equations of third order. Compared with the classical iteration algorithm, given by Saberi-Nadjafi and Tamamgar (2008), Abbasbandy and Shivanian (2009) and Abbasbandy et al. (2008, 2009), the present one is much more concise and effective and no unnecessary repeated calculation is needed as shown in the given examples. The elegance of variational iteration algorithm-II can be attributed to its simplistic approach in seeking the analytical solution of the system

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of nonlinear integro-differential equations, given as:

$$\begin{aligned}
 f_1^{(m)}(x) &= G_1(x, f_2(x), \dots, f_2^{(m)}(x), f_3(x), \dots, f_3^{(m)}(x), \dots, f_n(x), \dots, f_n^{(m)}(x)) \\
 &+ \int_0^x H_1(x, t, f_1(t), \dots, f_1^{(m)}(t), \dots, f_n(t), \dots, f_n^{(m)}(t)) dt \\
 f_2^{(m)}(x) &= G_2(x, f_1(x), \dots, f_1^{(m)}(x), f_3(x), \dots, f_3^{(m)}(x), \dots, f_n(x), \dots, f_n^{(m)}(x)) \\
 &+ \int_0^x H_2(x, t, f_1(t), \dots, f_1^{(m)}(t), \dots, f_n(t), \dots, f_n^{(m)}(t)) dt \quad (1) \\
 f_n^{(m)}(x) &= G_n(x, f_1(x), \dots, f_1^{(m)}(x), f_2(x), \dots, f_2^{(m)}(x), \dots, f_{n-1}(x), \dots, f_{n-1}^{(m)}(x)) \\
 &+ \int_0^x H_n(x, t, f_1(t), \dots, f_1^{(m)}(t), \dots, f_n(t), \dots, f_n^{(m)}(t)) dt
 \end{aligned}$$

with initial conditions $f_i^{(k)}(b_i) = c_k^i, k = 0, 1, 2, \dots, m-1, i = 1, 2, \dots, n.$ (2)

In system (1), $f_i(x)$ is the solution to be determined, m is the order of derivatives and $G_i, H_i, i = 1, 2, \dots, n$ are known linear or nonlinear functions.

VARIATIONAL ITERATION ALGORITHM-II

According to VIM, Equations [7 to 10], we can construct a correction functional of system (1) as:

$$\begin{aligned}
 f_{i,k+1}(x) &= f_{i,k}(x) + \int_0^x \lambda_i(s) [f_{i,k}^{(m)}(s) - G_n(s, \tilde{f}_{1,k}(x), \\
 &\dots, \tilde{f}_{1,k}^{(m)}(s), \dots, \tilde{f}_{i-1,k}(s), \dots, \tilde{f}_{i-1,k}^{(m)}(s), \tilde{f}_{i+1,k}(s), \dots, \tilde{f}_{i+1,k}^{(m)}(s) \\
 &\dots, \tilde{f}_{n,k}(s), \dots, \tilde{f}_{n,k}^{(m)}(s)) - \int_0^s H_n(s, t, \tilde{f}_1(t), \\
 &\dots, \tilde{f}_1^{(m)}(t), \dots, \tilde{f}_n(t), \dots, \tilde{f}_n^{(m)}(t)) dt] ds \quad (3)
 \end{aligned}$$

for $i = 1, 2, 3, \dots, n$ and $\lambda_i(x)$ are Lagrange multiplier and $f_{i,0}(x), i = 1, 2, 3, \dots, n$ are initial approximations which satisfy the initial conditions (2), \tilde{f}_n are considered to be restricted variations, that is, $\delta \tilde{f}_n = 0$. Making the

correction functional (3) stationary, and noticing that $\delta \tilde{f}_n = 0$,

$$\begin{aligned}
 \delta f_{i,k+1}(x) &= \delta f_{i,k}(x) + \delta \int_0^x \lambda_i(s) [f_{i,k}^{(m)}(s) - G_n(s, \tilde{f}_{1,k}(x), \\
 &\dots, \tilde{f}_{1,k}^{(m)}(s), \dots, \tilde{f}_{i-1,k}(s), \dots, \tilde{f}_{i-1,k}^{(m)}(s), \tilde{f}_{i+1,k}(s), \dots, \tilde{f}_{i+1,k}^{(m)}(s) \\
 &\dots, \tilde{f}_{n,k}(s), \dots, \tilde{f}_{n,k}^{(m)}(s)) - \int_0^s H_n(s, t, \tilde{f}_1(t), \\
 &\dots, \tilde{f}_1^{(m)}(t), \dots, \tilde{f}_n(t), \dots, \tilde{f}_n^{(m)}(t)) dt] ds = 0
 \end{aligned}$$

or

$$\delta f_{i,k+1}(x) = \delta f_{i,k}(x) + \delta \int_0^x \lambda_i(s) f_{i,k}^{(m)}(s) ds = 0$$

or

$$\begin{aligned}
 \delta f_{i,k+1}(x) &= \delta f_{i,k}(x) + \lambda_i(s) \delta f_{i,k}^{(m-1)}(s) \Big|_{s=x} \\
 &+ (-1)^1 \lambda_i^{(1)}(s) \delta f_{i,k}^{(m-2)}(s) \Big|_{s=x} + \dots + \\
 &(-1)^{m-1} \lambda_i^{(m-1)}(s) \delta f_{i,k}(s) \Big|_{s=x} + (-1)^m \int_0^x \lambda_i^{(m)}(s) \delta f_{i,k}(s) ds = 0
 \end{aligned}$$

for $i = 1, 2, 3, \dots, n$, yields the following stationary conditions:

$$\begin{aligned}
 \delta f_{i,k}(s) &: \lambda_i^{(m)}(s) = 0, \\
 \delta f_{i,k}(s) &: 1 + (-1)^m \lambda_i^{(m-1)}(s) \Big|_{s=x} = 0, \\
 \delta f_{i,k}^{(m-j-1)}(s) &: \lambda_i^{(j)}(s) \Big|_{s=x} = 0, \quad j = 1, 2, 3, \dots, m-2, \\
 &i = 1, 2, 3, \dots, n \\
 \delta f_{i,k}^{(m-1)}(s) &: \lambda_i(s) \Big|_{s=x} = 0.
 \end{aligned}$$

The Lagrange multiplier can be identified as:

$$\lambda_i(s) = \frac{(-1)^m}{(m-1)!} (s-x)^{m-1},$$

However, using the new concept of VIM, (He et al., 2010; He and Wu, 2007), we suggest an alternative iteration formula:

$$\begin{aligned}
 f_{i,k+1}(x) &= f_{i,0}(x) - \frac{(-1)^m}{(m-1)!} \int_0^x (s-x)^{m-1} [G_n(s, \tilde{f}_{1,k}(x), \\
 &\dots, \tilde{f}_{1,k}^{(m)}(s), \dots, \tilde{f}_{i-1,k}(s), \dots, \tilde{f}_{i-1,k}^{(m)}(s), \tilde{f}_{i+1,k}(s), \dots, \tilde{f}_{i+1,k}^{(m)}(s) \\
 &\dots, \tilde{f}_{n,k}(s), \dots, \tilde{f}_{n,k}^{(m)}(s)) + \int_0^s H_n(s,t, \tilde{f}_1(t), \\
 &\dots, \tilde{f}_1^{(m)}(t), \dots, \tilde{f}_n(t), \dots, \tilde{f}_n^{(m)}(t)) dt] ds, \tag{4}
 \end{aligned}$$

for $i = 1, 2, 3, \dots, n$

In order to elucidate the solution procedure of the variational iteration algorithm-II, we consider the system of third order non-linear integro-differential equations.

Example 1. Consider the following 3rd order non-linear integro-differential system.

$$\begin{aligned}
 f'''(x) &= e^x + \frac{1}{2} - x - \frac{1}{2} g''(x) + \frac{1}{2} \int_0^x (f'(t) + g'(t)) dt \\
 g'''(x) &= -e^x + 1 - f''(x) + \frac{1}{4} \int_0^x (f'(t) - g'(t)) dt
 \end{aligned} \tag{5}$$

with initial conditions,

$$\begin{aligned}
 f(0) &= 1, f'(0) = 2 \\
 f''(0) &= 1 \text{ and } g(0) = -1, g'(0) = 0, g''(0) = -1
 \end{aligned} \tag{6}$$

The exact solution of the problem is:

$$f(x) = x + e^x, \quad g(x) = x - e^x \tag{7}$$

According to the classical variational iteration method, (Saberi-Nadjafi and Tamamgar 2008; Abbasbandy and Shivanian, 2009; Abbasbandy et al., 2008; Sweilam, 2007), the correction functional can be written in the following form

$$\begin{aligned}
 f_{(n+1)}(x) &= f_{(n)}(x) + \int_0^x \lambda_1(s) \left(\tilde{f}_{(n)}'''(s) - e^s \right. \\
 &\left. - \frac{1}{2} + s + \frac{1}{2} \tilde{g}_{(n)}''(s) - \frac{1}{2} \int_0^s (\tilde{f}_{(n)}'(t) + \tilde{g}_{(n)}'(t)) dt \right) ds \\
 g_{(n+1)}(x) &= g_{(n)}(x) + \int_0^x \lambda_2(s) \left(\tilde{g}_{(n)}'''(s) + e^s \right. \\
 &\left. - 1 + \tilde{f}_{(n)}''(s) - \frac{1}{4} \int_0^s (\tilde{f}_{(n)}'(t) - \tilde{g}_{(n)}'(t)) dt \right) ds
 \end{aligned} \tag{8}$$

Where $\lambda_1(s)$ and $\lambda_2(s)$ are Lagrange multiplier, and $\tilde{f}(s)$ and $\tilde{g}(s)$ are restricted variations. Imposing the stationary condition ($\delta f_{(n+1)}(x) = 0$ and $\delta g_{(n+1)}(x) = 0$) on the correctional functional, the Lagrange multiplier can be readily identified in the following form

$$\lambda_1(s) = \lambda_2(s) = -\frac{1}{2}(s-x)^2 \tag{9}$$

As a result, we obtain the following iteration formula:

$$\begin{aligned}
 f_{(n+1)}(x) &= f_{(n)}(x) + \int_0^x \left(-\frac{1}{2}(s-x)^2 \right) \left(f_{(n)}'''(s) - e^s \right. \\
 &\left. - \frac{1}{2} + s + \frac{1}{2} g_{(n)}''(s) - \frac{1}{2} \int_0^s (f_{(n)}'(t) + g_{(n)}'(t)) dt \right) ds \\
 g_{(n+1)}(x) &= g_{(n)}(x) + \int_0^x \left(-\frac{1}{2}(s-x)^2 \right) \left(g_{(n)}'''(s) + 1 \right. \\
 &\left. - f_{(n)}''(s) + \frac{1}{4} \int_0^s (f_{(n)}'(t) - g_{(n)}'(t)) dt \right) ds
 \end{aligned} \tag{10}$$

we choose the initial solutions in the forms

$$f_0(x) = 1 + 2x + \frac{1}{2}x^2 \quad \text{and} \quad g_0(x) = -1 - \frac{1}{2}x^2 \tag{11}$$

This satisfies the initial conditions in Equation 6.

By the iteration formula (Equation 10) and the initial solution (Equation 11), we have the following first order approximation solutions

$$\begin{aligned}
 f_{(1)}(x) &= f_{(0)}(x) + \int_0^x \left(-\frac{1}{2}(s-x)^2 \right) \left(f_{(0)}'''(s) - e^s \right. \\
 &\left. - \frac{1}{2} + s + \frac{1}{2} g_{(0)}''(s) - \frac{1}{2} \int_0^s (f_{(0)}'(t) + g_{(0)}'(t)) dt \right) ds \\
 &= e^x + x + \frac{x^4}{24} + \frac{x^5}{60} + \frac{x^6}{360}
 \end{aligned} \tag{12}$$

$$\begin{aligned}
 g_{(1)}(x) &= g_{(0)}(x) + \int_0^x \left(-\frac{1}{2}(s-x)^2 \right) \left(g_{(0)}'''(s) + 1 \right. \\
 &\quad \left. - f_{(0)}''(s) + \frac{1}{4} \int_0^s (f_{(0)}'(t) - g_{(0)}'(t)) dt \right) ds \\
 &= -e^x + x + \frac{x^4}{24} + \frac{x^5}{120} \tag{13}
 \end{aligned}$$

Continuing in this manner, we obtain the other components of $f_{(n+1)}(x)$ and $g_{(n+1)}(x)$

$$\begin{aligned}
 f_{(2)}(x) &= 81 - 80e^x + 29x + 53e^x x + \frac{7x^2}{2} - 16e^x x^2 + \frac{x^3}{6} + \frac{17e^x x^3}{6} - \frac{7e^x x^4}{24} + \frac{e^x x^5}{60} \\
 &\quad - \frac{x^7}{5040} - \frac{x^8}{5760} - \frac{x^9}{45360} + \frac{x^{10}}{181440} + \frac{x^{11}}{380160} + \frac{41x^{12}}{68428800} + \frac{x^{13}}{12355200} + \frac{x^{14}}{172972800} \\
 g_{(2)}(x) &= \frac{47}{2} - \frac{49e^x}{2} + \frac{19x}{2} + 15e^x x + \frac{5x^2}{4} - \frac{9e^x x^2}{2} + \frac{x^3}{12} + \frac{5e^x x^3}{6} - \frac{5e^x x^4}{48} - \frac{x^5}{120} + \frac{e^x x^5}{120} \\
 &\quad - \frac{x^6}{360} - \frac{x^7}{2520} + \frac{x^8}{80640} + \frac{x^9}{362880} + \frac{x^{11}}{2280960} + \frac{31x^{12}}{136857600} + \frac{x^{13}}{24710400} + \frac{x^{14}}{345945600} \\
 f_{(3)}(x) &= -\frac{233531999983989}{16384} + \frac{28507321371e^x}{2} + \frac{23329141e^{2x}}{16384} - \frac{16657555191939x}{8192} \\
 &\quad - 12220268945e^x x - \frac{6629497e^{2x} x}{4096} - \frac{831700590567x^2}{8192} + \frac{10389928139e^x x^2}{2} + \dots \\
 g_{(3)}(x) &= -\frac{157150177236793}{32768} + \frac{19183368817e^x}{4} + \frac{19855161e^{2x}}{32768} - \frac{11312112683627x}{16384} \\
 &\quad - \frac{16421624645e^x x}{4} - \frac{5650023e^{2x} x}{8192} - \frac{570608576875x^2}{16384} + 1742312038e^x x^2 + \dots \tag{14}
 \end{aligned}$$

According to VIM-II (He et al., 2010; He and Wu, 2007), the iteration formula for Equation (5) is

$$\begin{aligned}
 f_{(n+1)}(x) &= f_{(0)}(x) + \int_0^x \left(\frac{1}{2}(s-x)^2 \right) \left(e^s + \frac{1}{2} - s \right. \\
 &\quad \left. - \frac{1}{2} g_{(n)}''(s) + \frac{1}{2} \int_0^s (f_{(n)}'(t) + g_{(n)}'(t)) dt \right) ds \\
 g_{(n+1)}(x) &= g_{(0)}(x) + \int_0^x \left(\frac{1}{2}(s-x)^2 \right) \left(e^s - 1 + f_{(n)}''(s) \right. \\
 &\quad \left. - \frac{1}{4} \int_0^s (f_{(n)}'(t) - g_{(n)}'(t)) dt \right) ds \tag{15}
 \end{aligned}$$

with the starting of the initial approximation $f_0(x) = 1 + 2x + \frac{1}{2}x^2$ and $g_0(x) = -1 - \frac{1}{2}x^2$, we have

$$\begin{aligned}
 f_0(x) &= 1 + 2x + \frac{1}{2}x^2 \\
 g_0(x) &= -1 - \frac{1}{2}x^2 \\
 f_1(x) &= e^x + x + \frac{x^4}{24} + \frac{x^5}{60} + \frac{x^6}{360} \\
 g_1(x) &= -e^x + x + \frac{x^4}{24} + \frac{x^5}{120} \\
 f_{(2)}(x) &= 81 - 80e^x + 29x + 53e^x x + \frac{7x^2}{2} - 16e^x x^2 + \frac{x^3}{6} + \frac{17e^x x^3}{6} - \frac{7e^x x^4}{24} + \frac{e^x x^5}{60} \\
 &\quad - \frac{x^7}{5040} - \frac{x^8}{5760} - \frac{x^9}{45360} + \frac{x^{10}}{181440} + \frac{x^{11}}{380160} + \frac{41x^{12}}{68428800} + \frac{x^{13}}{12355200} + \frac{x^{14}}{172972800} \\
 g_{(2)}(x) &= \frac{47}{2} - \frac{49e^x}{2} + \frac{19x}{2} + 15e^x x + \frac{5x^2}{4} - \frac{9e^x x^2}{2} + \frac{x^3}{12} + \frac{5e^x x^3}{6} - \frac{5e^x x^4}{48} - \frac{x^5}{120} + \frac{e^x x^5}{120} \\
 &\quad - \frac{x^6}{360} - \frac{x^7}{2520} + \frac{x^8}{80640} + \frac{x^9}{362880} + \frac{x^{11}}{2280960} + \frac{31x^{12}}{136857600} + \frac{x^{13}}{24710400} + \frac{x^{14}}{345945600} \\
 f_{(3)}(x) &= -\frac{233531999983989}{16384} + \frac{28507321371e^x}{2} + \frac{23329141e^{2x}}{16384} - \frac{16657555191939x}{8192} \\
 &\quad - 12220268945e^x x - \frac{6629497e^{2x} x}{4096} - \frac{831700590567x^2}{8192} + \frac{10389928139e^x x^2}{2} + \dots \\
 g_{(3)}(x) &= -\frac{157150177236793}{32768} + \frac{19183368817e^x}{4} + \frac{19855161e^{2x}}{32768} - \frac{11312112683627x}{16384} \\
 &\quad - \frac{16421624645e^x x}{4} - \frac{5650023e^{2x} x}{8192} - \frac{570608576875x^2}{16384} + 1742312038e^x x^2 + \dots \tag{16}
 \end{aligned}$$

and so on. In a similar manner, the rest of the components can be obtained by using Equation 15.

Example 2. Consider the following 3rd order non-linear integro-differential system.

$$\begin{aligned}
 f'''(x) &= 1 - \frac{x}{2} - \frac{1}{2} f(x) g(x) + \int_0^x ((x-t)g(t) + f'(t)) dt \\
 g'''(x) &= \frac{x}{2} + \frac{1}{2} f(x) g(x) + \int_0^x ((x-t)f(t) - g'(t)) dt \tag{17}
 \end{aligned}$$

with initial conditions

$$\begin{aligned}
 f(0) &= 0, f'(0) = 1, f''(0) = 0 \text{ and } g(0) = 1, g'(0) = 0, \\
 g''(0) &= 1 \tag{18}
 \end{aligned}$$

The exact solution of the problem is

$$f(x) = \text{Sinh}(x), \quad g(x) = \text{Cosh}(x) \tag{19}$$

Using Equation 4 and by selecting $f_0(x) = x$ and $g_0(x) = 1 + \frac{1}{2}x^2$, we obtain

$$\begin{aligned}
 f_{(1)}(x) &= x + \int_0^x \left(\frac{1}{2}(s-x)^2 \right) \left(1 - \frac{s}{2} - \frac{1}{2}f_{(0)}(s)g_{(0)}(s) + \int_0^s ((x-t)g_{(0)}(t) + f_{(0)}'(t))dt \right) ds \\
 &= x + \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^6}{480} + \frac{x^7}{5040} \\
 g_{(1)}(x) &= 1 + \frac{x^2}{2} + \int_0^x \left(\frac{1}{2}(s-x)^2 \right) \left(\frac{s}{2} + \frac{1}{2}f_{(0)}(s)g_{(0)}(s) + \int_0^s ((x-t)f_{(0)}(t) - g_{(0)}'(t))dt \right) ds \\
 &= 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{1440}
 \end{aligned}
 \tag{20}$$

Similarly, one can obtain the second-order, third-order and fourth-order approximate solutions given below

$$\begin{aligned}
 f_{(2)}(x) &= x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} - \frac{x^9}{290304} + \frac{x^{10}}{2073600} - \frac{83x^{11}}{79833600} + \frac{x^{12}}{19958400} \\
 &\quad - \frac{7x^{13}}{197683200} + \frac{157x^{14}}{19372953600} - \frac{x^{15}}{1257984000} + \frac{x^{16}}{42268262400} \\
 g_{(2)}(x) &= 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^9}{483840} + \frac{x^{10}}{691200} - \frac{x^{11}}{1774080} + \frac{47x^{12}}{479001600} \\
 &\quad - \frac{x^{13}}{39536640} + \frac{29x^{14}}{11623772160} - \frac{x^{15}}{3773952000} + \frac{x^{16}}{48771072000} \\
 f_{(3)}(x) &= x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \frac{x^9}{362880} + \frac{x^{11}}{39916800} - \frac{x^{12}}{141926400} + \frac{59x^{13}}{49816166400} \\
 &\quad - \frac{457x^{14}}{174356582400} + \frac{787x^{15}}{2615348736000} - \frac{1609x^{16}}{5072191488000} + \dots \\
 g_{(3)}(x) &= 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \frac{x^8}{40320} + \frac{x^{10}}{3628800} - \frac{x^{12}}{3832012800} + \frac{17x^{13}}{9963233280} \\
 &\quad - \frac{73x^{14}}{43589145600} + \frac{1193x^{15}}{2615348736000} - \frac{37451x^{16}}{167382319104000} + \dots
 \end{aligned}
 \tag{21}$$

Example 3. Consider the following 3rd order non-linear integro-differential system.

$$\begin{aligned}
 u'''(x) &= -x - v''(x) + \int_0^x (u'^2(t) + v'^2(t))dt \\
 v'''(x) &= -x^2 + u''(x) + \int_0^x x(v''^2(t) + u(t)v'(t))dt
 \end{aligned}
 \tag{22}$$

with initial conditions

$$u(0) = 1, u'(0) = 0, u''(0) = -1 \text{ and } v(0) = 0, v'(0) = 1, v''(0) = 0$$

The exact solution is

$$u(x) = \text{Cos}(x), v(x) = \text{Sin}(x)
 \tag{23}$$

Its iteration formulation is suggested according to He et al. (2010), and He and Wu (2007).

$$\begin{aligned}
 u_{(n+1)}(x) &= u_{(n)}(x) - \int_0^x \left(\frac{1}{2}(s-x)^2 \right) \left(s + v_{(n)}''(s) - \int_0^s (u_{(n)}'(t) + v_{(n)}'(t))dt \right) ds \\
 v_{(n+1)}(x) &= v_{(n)}(x) - \int_0^x \left(\frac{1}{2}(s-x)^2 \right) \left(s^2 - u_{(n)}''(s) - s \int_0^s (v_{(n)}''(t) + u_{(n)}(t)v_{(n)}'(t))dt \right) ds
 \end{aligned}
 \tag{24}$$

Beginning with

$$u_0(x) = 1 - \frac{x^2}{2} \text{ and } v_0(x) = x
 \tag{25}$$

We have

$$\begin{aligned}
 u_{(1)}(x) &= u_{(0)}(x) - \int_0^x \left(\frac{1}{2}(s-x)^2 \right) \left(s + v_{(0)}''(s) - \int_0^s (u_{(0)}'(t) + v_{(0)}'(t))dt \right) ds \\
 &= 1 - \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{18} \\
 v_{(1)}(x) &= v_{(0)}(x) - \int_0^x \left(\frac{1}{2}(s-x)^2 \right) \left(s^2 - u_{(0)}''(s) - s \int_0^s (v_{(0)}''(t) + u_{(0)}(t)v_{(0)}'(t))dt \right) ds \\
 &= x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{144} \\
 u_{(2)}(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{6} - \frac{x^6}{240} - \frac{91x^8}{5760} - \frac{23x^{10}}{1512} + \frac{229x^{12}}{31104} + \frac{19x^{14}}{12672} + \frac{49x^{16}}{1617408} \\
 v_{(2)}(x) &= x - \frac{x^3}{6} + \frac{x^5}{20} + \frac{x^7}{126} - \frac{x^9}{240} + \frac{103x^{11}}{24192} - \frac{161x^{13}}{124416} + \frac{x^{15}}{3072} - \frac{7x^{17}}{808704} \\
 u_{(3)}(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{6} - \frac{x^6}{120} - \frac{103x^8}{5040} + \frac{1343x^{10}}{151200} + \frac{10663x^{12}}{4354560} + \frac{2187331x^{14}}{479001600} \\
 &\quad - \frac{164055193x^{16}}{27172454400} + \frac{226161763x^{18}}{140107968000} + \frac{42090421x^{20}}{79394515200} - \dots \\
 v_{(3)}(x) &= x - \frac{x^3}{6} + \frac{7x^5}{120} - \frac{x^7}{1680} - \frac{667x^9}{51840} - \frac{443x^{11}}{1330560} + \frac{101213x^{13}}{16174080} - \frac{322403x^{15}}{63866880} \\
 &\quad + \frac{584431x^{17}}{164975616} - \frac{7415129x^{19}}{21555072000} - \dots
 \end{aligned}
 \tag{26}$$

CONCLUSIONS

In this paper, we have applied an effective variational iteration algorithm-II for solving third order nonlinear integro-differential equations. The aim of this paper is two folds. Firstly, we reveal that the new iteration formulations are much more effective; secondly, the new algorithm is of mathematical significance and of application features. The solutions presented by variational iteration algorithm-II in this paper show excellent agreement with the exact solutions. The graphical analysis reported here further shows confidence on VIM-II.

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