Synthesis of hyperboloid gear drives: Controlling of the singularity on the active tooth surfaces

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The study illustrates the mathematical model, applied to the synthesis of hyperboloid gear drives with taking into account the singularity appearance on the contacting active tooth surfaces. A kinematic approach for registration of the singularity on the tooth surfaces of the synthesized spatial transmissions is shown. The analytical dependencies applicable for defining and controlling of the singularities on the conjugates active surfaces are illustrated.

Key words: Hyperboloid gear drives, kinematic synthesis, mathematical modeling, region of mesh, singularity.

INTRODUCTION

Mechanisms, in which the rotations transformation between non-coplanar axes is realized by set of high kinematic joints, which elements comes in and goes out of tangential contact, by following a certain logical sequence, are known in the literature (Abadjiev, 2007; Abadjiev and Abadjieva, 2016 a, b; Litvin, 1968, 1989, 1994) as hyperboloid gear drives (spatial gear mechanisms with crossed axes of rotation). The rotations transformation is realized as a result of the normal forces, which acts in the places of tangential contact of the elements of high kinematic joints.

It is known, that the design of mechanical multibody systems is a complex task, which main stage is known in Applied Mechanics under the name Synthesis of Mechanisms. In the most general case, the synthesis of the mechanisms includes the following two main tasks (Abadjiev, 2007):

(1) Synthesis of the structure of the designed mechanism by carrying out a structural synthesis.
(2) Design of the kinematic scheme of the mechanism. This task is known as kinematic synthesis of the mechanism. Through its solution the determination of the constant geometrical parameters of the chosen structure of the mechanism is achieved. These parameters satisfy its preliminary defined kinematic characteristics and related with them specific geometric features of the mechanism.

In the case of synthesis of hyperboloid gear mechanisms (spatial gear drives with crossed axes of rotation), it is not necessary to solve the problem of structural synthesis. The spatial rotations transformation, as a rule, is realized by three-links gear mechanisms, whose movable links rotations axes are crossed in the space. The motions...
transformation and transmission of mechanical energy is accomplished through a system of high kinematic joints.

Therefore, the synthesis of different type hyperboloid gear mechanisms is reduced to finding out a solution of the kinematic synthesis task. The registration and control of the singularity of the conjugate active tooth surfaces is accomplished through the developed adequate mathematical model for the synthesis of that class gear drives.

Singularity is a characteristic of a great importance for the processes of generation and conjugate action of the active tooth surfaces of the synthesized hyperboloid gear drives. The reason for this is that it defines the undercutting points, the increased friction, low lubrication, pitting, etc., on the contacting tooth surfaces (Abadjiev, 2007; Abadjiev and Abadjieva, 2016a). This characteristic is often ignored from the designers, due to the lack of knowledge of its existence. Its localization and control are complicated processes, which require a development of adequate mathematical approaches.

**METHODOLOGY**

The aim of the task of the kinematic synthesis of every mechanical system is the definition of optimal values of the so-called independent parameters (parameters of the synthesis). The determination of the set of parameters of the synthesis is by complying of two sets of conditions: basic and additional ones (Abadjiev, 2007).

In order to synthesize a mechanism that satisfies some preliminary given characteristics, it is necessary to combine many contradictory conditions, relating to the purpose of the gear mechanism, to the technology of its manufacturing, to the conditions of its exploitation and so on. One of these conditions is a basic one. For the gear mechanisms, including the hyperboloid gear drives, the basic condition of the synthesis has a kinematic character. A condition of this type is the realization of the transfer function of the mechanism. For prevailing in practice cases of synthesis of gear mechanisms that condition is the gear ratio (ratio of the values \( \omega_1, (i = 1, 2) \) of the angular velocity vectors):

\[
i_{12} = \frac{\omega_1}{\omega_2} = \text{const.}
\]  

(1)

In this case, for the synthesis the optimum value of deviation between actual and theoretical given gear ratio is pursued.

In other cases, the synthesis of gear drives can be subjected to the ensuring of the optimal efficiency of motions transformation by reaching a preliminary given value of the efficiency, which in this case is a basic condition.

All of the others conditions are additional ones. Into this group, the restrictions regarding the size of the movable links and their strength loading, the control of the singularity of the conjugate active tooth surfaces (elements of high kinematic joints), etc., can be included.

In most cases of the practice, the main purpose of every gear mechanism is to realize with necessary accuracy, at minimal losses of the transmitted mechanical energy and optimal strength characteristics, the preliminary given law of motions transformation.

The gear drives with crossed axes of rotations are characterized with the presence of large number of free parameters. When suitable combinations among them were searched for, the possibility for the synthesis of these class transmissions to obtain the desired optimum combinations of technological and exploitation characteristics were created.

The said up to now, determines that applied by the authors of this study, kinematic character of the approach to the synthesis and respectively the kinematic character of the created mathematical models. This approach, applied by authors, for registration of the singularity in the mesh region of the synthesized hyperboloid gear drives is a kinematically oriented. The control (registration and elimination) of the singularity of the conjugate active tooth surfaces (elements of high kinematic joints), depending on the designation of the designed transmission can be treated as the basic condition of the synthesis.

**Mathematical model for synthesis upon a pitch contact point**

The mathematical model for synthesis upon a pitch contact point is based on the assumption, that the necessary quality characteristics, defining a concrete exploitation and technological requirements to the active tooth surfaces, are guaranteed for only one contact point \( P \) (for its close vicinity, respectively) of the active tooth surfaces \( \Sigma_1 \) and \( \Sigma_2 \) (Figure 1) (Abadjiev, 2007; Abadjiev and Abadjieva, 2016a). The model is applicable to the synthesis of spatial gear mechanisms, not only with linear contact but with the point contact.

According to it, the common contact point \( P \) of the conjugate tooth surfaces \( \Sigma_1 \) and \( \Sigma_2 \) is a common contact point of the pair of two circles \( H^c_i \) (\( i = 1, 2 \)) - pitch circles \( (H^c_i : H^c_j) \), that is, the point \( P \) is a pitch contact point. Plane \( T_m \), that includes the tangents to \( H^c_i \) (\( i = 1, 2 \)) at the point \( P \), is a pitch plane, and \( m-m \) is the pitch normal to \( T_m \) at the point \( P \). Figure 1 illustrates a case of mutual position of the pitch circles in the fixed space, corresponding to the traditional constructions of hyperboloid gear transmissions with an externally mating active tooth surfaces.

The diameters \( d_i \) (\( i = 1, 2 \)) of the \( H^c_i \) (\( i = 1, 2 \)), together with parameters \( a_i, \theta_i, \delta_i \) (\( i = 1, 2 \)), \( \delta \) and \( a \), determine the structure type and its dimensions and shape, as well as the dimensions and mutual positions of the active links in the fixed space (in coordinate systems \( S_i(O_i, x_i, y_i, z_i) \), (\( i = 1, 2 \))). These dimensions are also related to defining the longitudinal and crossed orientations of the active tooth surfaces \( \Sigma_1 \) (\( i = 1, 2 \)) at the pitch contact point, as well as their sizes.

In other words, the pair circles \( (H^c_i : H^c_j) \) are directly related to determining the pitch of the teeth and the tooth module of the designed gear system, respectively. The parameters \( d_i, \delta_i \) (\( i = 1, 2 \)) define the sizes of the reference co-axial rotation surfaces, that is, the dimensions of the gear blanks depend on them. The aforementioned parameters are used in sizing of the bearing of the synthesized gear drive.

Hence, from what is said up to here, the mathematical model for synthesis upon a pitch contact point ensures the algorithmic solution of two type basic tasks (Abadjiev, 2007; Abadjiev and Abadjieva, 2016a):

1. Synthesis of the pitch configurations;
2. Synthesis of the active tooth surfaces.
Figure 1. Geometric-kinematic interpretation of the model for synthesis upon pitch contact point: $H_i^j (i=1, 2)$ - pitch circles; $T_m$ - pitch surface; $m_m$ - pitch normal line to $T_m$ at point $P$; $\Sigma_i$ $(i=1, 2)$ - contacting at $p$ active surfaces.

When these tasks are solved together, the necessary preliminary defined geometric characteristics of the synthesized gear mechanisms in close vicinity of the pitch contact point are ensured. In conclusion, it should be noted that the approach, described here, for the synthesis of spatial gear drives is based on the following kinematic condition:

The relative velocity vector: $V_{12} = V_1 - V_2 = \vec{w}_1 \times \vec{O}_1P - \vec{w}_2 \times \vec{O}_2P$ ($V_i (i=1,2)$ - circumferential velocity vectors) at the pitch contact point $P$ has to lie in the pitch plane $T_m$ and in the common tangential plane $T_n$ of the contacting at $P$ tooth surfaces $\Sigma_1$ and $\Sigma_2$, while being oriented towards the direction of the common tangent line to the longitudinal lines of the active tooth surfaces $\Sigma_i$ $(i=1, 2)$.

It should be emphasized that the aimed approach to the basic synthesis upon a pitch contact point, based on solving the mentioned tasks, is characterized in that the mathematical model and based on this developed algorithm there is a typical feature universal structure for all types of hyperboloid gear transmissions.

Methodology of the synthesis of the conjugate tooth surfaces of hyperboloid gear drives with linear contact

When hyperboloid gear drives with linear contact between their active surfaces are synthesized, it is necessary to control the quality of mating in the entire mesh region or in a fixed, by some reasons, zone. This approach to the synthesis task requires development of an adequate approach. Its common kinematic scheme is as shown in Figure 2 (Abadjiev, 2007).

On the basis of mathematical modeling for synthesis of spatial gears upon a region of mesh is the kinematical model of the surfaces of action.

Following the approach, we will note that the optimization process is essentially a determination of the optimal geometry and the limits of the mesh region as part of the surface of action (Figure 2). As mentioned, when a pair of gears with linear contact is synthesized, it is necessary to control their quality characteristics in whole mesh region. The surface of action will be defined through the active tooth surfaces of one of the movable links of the hyperboloid mechanism.

To illustrate this approach, let us accept that the $\Sigma_i$ - an active surface of one of the movable links $(i=1$ - pinion) of the three-link gear mechanism is known. The technology for the generation of the active tooth surfaces of these mechanisms is based on the second Olivier’s principle (Abadjiev, 2007; Litvin, 1968). Let $\Sigma_i$ is presented with its vector equation:

$$\vec{\rho}_{i,p} = \vec{\rho}_{i,p}(u, \vartheta),$$

(2)

where $\vec{\rho}_{i,p}$ is a radius vector of the point from $\Sigma_i$ in the
Figure 2. Geometric-kinematic interpretation of the model for synthesis upon mesh region: \( L_1 \) - longitudinal line of \( \Sigma_i \); \( D_{i1} \) - contact line between \( \Sigma_j \) (pinion) and \( \Sigma_j \) (crown); \( AS \) - action surface (the mentioned in the text coordinate systems are not illustrated here).

coordinate system \( S_p (O_p, x_p, y_p, z_p) \), fixed to the pinion; \( u, \vartheta \) parameters of \( \Sigma_j \), defined in \( S_p \).

Equation 2 describes the instrumental surfaces \( \Sigma_j \equiv \Sigma_j \), which generate the tooth surfaces \( \Sigma_2 \) of the second gear \( (i = 2) \). We accept that on \( \Sigma_j \) the bendings and interruptions are missing, that is, the condition is fulfilled:

\[
\overline{n}_{1,p} = \frac{\partial \overline{p}_{1,p}}{\partial u} \times \frac{\partial \overline{p}_{1,p}}{\partial \vartheta} \neq 0. \tag{3}
\]

Here \( \overline{n}_{1,p} \) is a normal vector at arbitrary point from \( \Sigma_j \), which is written in the coordinate system, \( S_p \) is of the form (Abadjiev, 2007; Litvin, 1968, 1989, 1994):

\[
\begin{align*}
n_{1,x_p} &= \frac{\partial y_p}{\partial u} \frac{\partial z_p}{\partial \vartheta} - \frac{\partial y_p}{\partial \vartheta} \frac{\partial z_p}{\partial u}, \\
n_{1,y_p} &= \frac{\partial z_p}{\partial u} \frac{\partial x_p}{\partial \vartheta} - \frac{\partial z_p}{\partial \vartheta} \frac{\partial x_p}{\partial u}, \\
n_{2,z_2} &= \frac{\partial x_p}{\partial u} \frac{\partial y_p}{\partial \vartheta} - \frac{\partial x_p}{\partial \vartheta} \frac{\partial y_p}{\partial u}. \tag{4}
\end{align*}
\]

In accordance with the kinematic approach to synthesis, the contact lines on the tooth surface \( \Sigma_j \) can be defined by direct application of the basic theorem of meshing (Abadjiev, 2007; Litvin, 1968, 1989, 1994), that is:

\[
\overline{p}_{1,p} = \overline{p}_{1,p} (u, \vartheta), \quad \overline{n}_{1,p} \overline{V}_{12,p} = 0, \tag{5}
\]

where \( \overline{V}_{12,p} \) is a relative velocity vector in arbitrary point from \( \Sigma_j \).
The co-ordinate form of Equation 5 is:

\[ x_p = x_p(u, \vartheta), \quad y_p = y_p(u, \vartheta), \]
\[ z_p = z_p(u, \vartheta), \]
\[ [V_{12,x}, V_{12,y}, V_{12,z}]^T = \]
\[ = L_p[V_{12,x}, V_{12,y}, V_{12,z}]^T, \]
\[ n_{i,x}V_{12,x} + n_{i,y}V_{12,y} + n_{i,z}V_{12,z} \equiv \]
\[ \equiv f_p(u, \vartheta, \varphi_p) = 0, \]  

where \( L_p \) is a 3×3 transformation matrix from the fixed co-ordinate system \( S \) (co-ordinate system fixed to the frame) into co-ordinate system \( S_p \) of the pinion; \( \varphi_1 \) — parameter of meshing. If the equations system 5 is written in co-ordinate system \( S(O, x, y, z) \), then the mesh region is obtained as a locus of the contact lines between \( \Sigma_1 \) and \( \Sigma_2 \) in the fixed space:

\[ x_p = x_p(u, \vartheta), \quad y_p = y_p(u, \vartheta), \]
\[ z_p = z_p(u, \vartheta), \]
\[ n_{i,x} = n_{i,x}(u, \vartheta), \quad n_{i,y} = n_{i,y}(u, \vartheta), \]
\[ n_{i,z} = n_{i,z}(u, \vartheta), \]
\[ [x \ y \ z \ t]^T = M_p[x_p \ y_p \ z_p \ t_p]^T, \]
\[ [n_{i,x} \ n_{i,y} \ n_{i,z}]^T = L_p[n_{i,x} \ n_{i,y} \ n_{i,z}]^T, \]
\[ n_{i,x}V_{12,x} + n_{i,y}V_{12,y} + n_{i,z}V_{12,z} \equiv \]
\[ \equiv f_p(u, \vartheta, \varphi_p) = 0. \]

Here \( M_p \) and \( L_p \) are respectively 4×4 and 3×3 transformation matrices from \( S_p \) into \( S \). Analogically, the contact lines on \( \Sigma_1 \) can be written in the co-ordinate system \( S \) of the second movable link \( i = 2 \).

Generally, the defined geometry, dimensions and location of the mesh region in the fixed space, as part of the action surface are optimal ones, if:

1. The singular points on it are registered and eliminated;
2. The orientation and placement of the contact lines on the mesh region are determined. This is realized in order to reach the maximum possible loading capacity and coefficient of efficiency.

**RESULTS AND DISCUSSION**

**Active tooth surfaces singularity registration**

On the questions dealing with the problems of singularity of conjugate tooth surfaces, studies of many scientists, independently and in collaboration have treated it (Litvin and Erihov, 1970; Litvin, 1968; Lagutin, 2000; Minkov, 1975; Nelson, 1961) and others are dedicated.

When mutual enveloping surfaces \( \Sigma_1 \) and \( \Sigma_2 \) are mated, for certain geometric and kinematic conditions, part of the contact points are transformed into node (singular) points, for which the following condition is fulfilled: relative velocity vector at one of the contacting tooth surfaces is the zero vector (Abadjiev, 2007):

\[ \bar{V}_{r,i} = \nabla_{\vartheta} \frac{d\vartheta}{dt} + \frac{\partial \vartheta}{\partial \vartheta} \frac{d\vartheta}{dt} = \bar{O}. \]  

The Equation 8 is correct both for \( \Sigma_1 \) and for \( \Sigma_2 \), since it relates to the relative velocity vector at a point from a concrete contact line, that is, \( \vartheta_i = constant \) and respectively \( \vartheta_2 = constant \) \( i = 1, 2 \) — meshing parameter. Depending on the behavior of the normal vector \( \bar{n}_i \) at the contact point of the mated tooth surfaces \( \Sigma_1 \) and \( \Sigma_2 \), two types of singular points can be defined:

1. **Singular points of first order (ordinary nodes):** For them it is fulfilled \( \bar{n}_i \neq \bar{O} \) and therefore, \( \bar{n}_{r,i} = \bar{O} \) (\( \bar{n}_{r,i} \) relative velocity vector at the tip of the normal vector \( \bar{n}_i \)).
   
   They are the points of contact or of intersection of the contact lines. Their existence leads to dry friction, high specific sliding, and decreasing of the hydrodynamic loading capacity.

2. **Singular points of second order (points of undercutting):** For them the condition \( \bar{n}_i = \bar{O} \) is fulfilled.

The undercutting leads to decrease of bending strength and contact strength of the generated teeth.

Further in the study, the character of two types undercutting points will be explained, in the context of kinematically conjugate tooth surfaces, whose synthesis is realized in accordance with the second principle (Abadjiev, 2007; Litvin, 1968). In other words, we will search vector and analytical dependencies, through which the conditions of undercutting point appearance in the mesh region, respectively on the mating tooth surfaces of the synthesized gear mechanism, are defined. And also we will search for possible approaches for their elimination in the process of synthesis in connection with that caused by the negative effects accompanying processes of instrumental and work gearing.

Let the enveloped surface \( \Sigma_1 \) be given in its own co-ordinate system \( S_p \) of \( O_p, x_p, y_p, z_p \) with equation:

\[ \bar{\rho}_{i,p} = \bar{\rho}_{i,p}(u, \vartheta). \]
And the defined with Equation 9 surface $\Sigma_j$ is the regular ones, that is, it contains only ordinary nodes and hence for all the point following the condition is fulfilled:

$$\vec{n}_{i,p} = \frac{\partial \vec{p}_{i,p}}{\partial u} \times \frac{\partial \vec{p}_{i,p}}{\partial \vartheta} \neq 0.$$  \hspace{1cm} (10)

After joining to Equation 9, the equation of meshing is

$$f_p (u, \vartheta, \varphi_j) = 0.$$  \hspace{1cm} (11)

The contact lines on surface $\Sigma_j$ (belonging to the first movable link-pinion) are obtained. They are contact lines between $\Sigma_j$ and $\Sigma_2$, for the different values of the meshing parameter, $\varphi_j$. If the equation set, consisting of Equations 9 and 11, is written in the fixed co-ordinate system $S(O, x, y, z)$, then the mesh region is obtained as a locus of the contact lines in the fixed space. When the same equations system is written in the co-ordinate system $S_g(O_g, x_g, y_g, z_g)$, static connected with second movable link (crown), the analytical expression of the conjugate tooth surface $\Sigma_2$ is obtained as a locus of the same contact lines, but in the non-fixed (rotating) space, defined by $S_g$.

As a rule, the analytical expression of $\Sigma_2$ is a complicated system of non-linear transcendent equations, which makes difficult the direct defining of the nodes’ placement on the surface $\Sigma_2$. This requires to find an indirect way for their registration on the mesh region and $\Sigma_2$, in particular by locating the placement of those regular points on $\Sigma_j$, which in the processes of instrumental and work meshing, will create possibilities for appearance of nodes: for instrumental meshing undercutting points and for work meshing ordinary nodes. This is achieved by applying the criteria of Equation 8, which is written, for the tooth surface $\Sigma_2$, is of the form:

$$\vec{V}_{r,2} = \frac{\partial \vec{p}_2}{\partial u} \frac{du}{dt} + \frac{\partial \vec{p}_2}{\partial \vartheta} \frac{d\vartheta}{dt} = \vec{0} \Rightarrow$$

$$\Rightarrow \frac{\partial \vec{p}_2}{\partial u} = \frac{\partial \vartheta}{\partial \vartheta} \frac{\partial \vec{p}_2}{\partial \vartheta}$$  \hspace{1cm} (12)

Condition 12 is fulfilled, if vectors $\frac{\partial \vec{p}_2}{\partial u}$ and $\frac{\partial \vec{p}_2}{\partial \vartheta}$ are collinear ones and they are zero vectors, that is, it is impossible to define a normal vector to $\Sigma_2$, at the concrete contact point ($\varphi_i = \text{constant, } i = 1, 2$):

$$\vec{n}_2 = \frac{\partial \vec{p}_2}{\partial u} \times \frac{\partial \vec{p}_2}{\partial \vartheta} = \vec{0}.$$  \hspace{1cm} (13)

Condition 12 is also fulfilled when

$$\frac{du}{dt} = \frac{d\vartheta}{dt} = 0.$$  \hspace{1cm} (14)

Without taking into account the behavior of the normal vector $\vec{n}_2$, that is, it is possible $\vec{n}_2 = \vec{0}$ and $\vec{n}_2 \neq \vec{0}$. On the other hand, it is known (Abadjiev, 2007; Abadjiev and Abadjieva, 2016 a, b; Litvin, 1968, 1989, 1994) that the relative velocity vector at the end (tip) of the normal vector in the contact point, as a point from $\Sigma_j$ is of the form:

$$\vec{\dot{n}}_r,1 = \vec{0}.$$  \hspace{1cm} (16)

Lets now clarify the meaning of the aforementioned condition. The existence of the surface $\Sigma_2$, for the given surface $\Sigma_j$, is possible if the equation of meshing (Equation 11) is defined. If it is consider as identity for every value of $\varphi_j$, it can be differentiated and it obtained the following dependence:

$$\frac{df}{dt} = \frac{\partial f}{\partial u} \frac{du}{dt} + \frac{\partial f}{\partial \vartheta} \frac{d\vartheta}{dt} + \frac{\partial f}{\partial \varphi_j} \frac{d\varphi_j}{dt} = 0.$$  \hspace{1cm} (17)

Comparing Equations 17 with 14, it is found that if Equation 14 is fulfilled, respectively Equation 16 and then Equation 17 are valid only if:

$$\frac{df}{\partial \varphi_j} = 0,$$  \hspace{1cm} because $\frac{d\varphi_j}{dt} \neq 0$.  \hspace{1cm} (18)

In other words, Equation 18, together with the condition (Equation 14) (respectively Equation 16) provide a validation of the criteria (Equation 8), when the contact point is a regular one, that is, for the common normal vector at it, the following condition is fulfilled:

$$\vec{n}_i \neq \vec{0}, (i = 1, 2)$$  \hspace{1cm} (19)
The realized analysis shows, that the criteria (Equation 8) located on the tooth surfaces (on the mesh region respectively) of all the nodes, considered as common points of the surfaces $\Sigma_1$ and $\Sigma_2$, in which when a $\vec{n}_1 \neq \vec{n}_2$ is defined, the normal vector $\vec{n}_2$ cannot be defined. These points are singular points of second order. The mentioned criteria locate nodes also, which are the points of tangential contact. These are common points of $\Sigma_1$ and $\Sigma_2$, for which the common normal vector at them exists $\vec{n}_2$, but at the same time the equality (Equation 18) is fulfilled. In this case the criteria, Equation 8 is satisfied, when there is a contact of regular surfaces $\Sigma_1$ and $\Sigma_2$. In fact, these contact points are called ordinary nodes (singular points of the first order).

Hence, the criteria (Equations 8 and 18) insure the registration of all nodes in the mesh region for every synthesized gear mechanism (Abadjiev, 2007; Litvin, 1968).

**Crossed orientation of the active tooth surfaces at the pitch contact point, when it is considered as an ordinary node**

Here, keeping the direction of the realized analysis, it will be clarified as that applied by authors analytical approach for the registration and elimination of the ordinary nodes on the meshed surfaces. In accordance with already determined conditions, defining one conjugate contact point, as an ordinary node, will be shown as the vector dependency connecting geometric and kinematic parameters of synthesized gear drives, which characterizes it as arbitrary ordinary node point.

Here and further when it is taken into account (Figure 3, symbols), the basic equation of meshing can be written
in the form:

\[ \mathbf{n}_i \cdot \mathbf{v}_{12} = \mathbf{n}_i \cdot [\mathbf{\alpha}_1 \times \mathbf{\rho}_1 - \mathbf{\alpha}_2 \times \mathbf{\rho}_2] = 0. \] (20)

Here \( \mathbf{n}_i \) is a normal vector to \( \Sigma_i \) in point \( P \) (it is not illustrated on Figure 3); \( \mathbf{v}_{12} \) - relative velocity vector in point \( P \); \( \mathbf{\alpha}_i \), \( (i = 1, 2) \) - angular velocities vectors of rotation of the links \( i = 1 \) and \( i = 2 \); \( \mathbf{\rho}_i \) - radius-vector of the contact point \( P \) \( (\rho_i \neq constant) \).

After differentiating the aforementioned equation and taking into account specific kinematic characteristics which make a regular contact point, to become an ordinary node, the following vector equation is obtained (Abadjiev, 2007; Litvin, 1968; Minkov, 1975):

\[ \mathbf{\alpha}_w (\mathbf{n}_{i,c} \times \mathbf{\rho}_i) \sin \delta - a_w \cos \delta \mathbf{n}_{i,c} = 0, \] (21)

where \( \mathbf{n}_{i,c} \) is a normal vector to the conjugate surfaces \( \Sigma_1 \) and \( \Sigma_2 \) at the ordinary node \( P \); \( \mathbf{\alpha}_w \) - vector – offset distance \( (a_w = constant) \); \( \delta = \angle(\mathbf{\alpha}_1, \mathbf{\alpha}_2) \) - crossed angle of the axes of rotation \( 1-1 \) and \( 2-2 \).

Without disturbing the community of arguments, in Equation 22 (obtained for \( \omega_i = constant \) it is substituted \( \omega_i = 1 \) rad/s and \( \omega_2 = i_{21} \) rad/s. Let it be accepted that for the pitch contact point, condition 22 is fulfilled, that is, the pitch contact point is an ordinary node.

If Equation 22 is written in the fixed co-ordinate system \( S(O, x, y, z) \), the angle \( \alpha_{cr} \) is obtained. It determines the orientations of \( \mathbf{n}_{i,c} \) in the pitch contact point to the pitch plane \( T_m \) (Figure 3).

Here \( \beta_i \) \( (i = 1, 2) \) are angles, defining the longitudinal orientation of the teeth at the pitch contact point \( P \) \( (\beta_i \neq 0) \); \( a_i \), \( r_i \), \( \theta_i \) \( (i = 1, 2) \) - cylindrical coordinates of the point \( P \); \( \delta_i \) \( (i = 1, 2) \) - angles determining the orientation of the surfaces,

\[ \alpha_{cr} = \arctan\{\sin(\pm \beta_i) \{r_i \cos \delta_i + a_i \cot \delta(A) + a_i(B)\}/(C)\}, \]
\[ A = \cot(\pm \beta_i) + \tan \theta_i \sin \delta_i, \]
\[ B = \cot(\pm \beta_i) \tan \theta_i - \sin \delta_i, \]
\[ C = r_i \sin \delta_i + a_i \cos \delta_i - a_i \cot \delta \tan \theta_i \cos \delta_i, \] (22)

containing the pitch circles to the common normal \( m-m \) at \( T_m \) in point \( P \). The expression (Equation 22) is written for the case, illustrated in Figure 3, when the surfaces \( \Sigma_1 \) and \( \Sigma_2 \) (contacting at the pitch contact point teeth \( P \)) have a different directions longitudinal lines \( L_1 \) and \( L_2 \). The signs are upper for \( i = 1 \), and lower – for the \( i = 2 \). The values of the angle \( \alpha_{cr} \) affect the size of the normal angle of the profile in the pitch contact point and through the main kinematic and strength characteristics onto the quality of the synthesized gear mechanism. The consideration of \( \alpha_{cr} \), when the synthesis is realized, is the reason for non-symmetrical tooth profile of the teeth of hyperboloid gear drives with face mating gears.

**Conclusion**

The kinematic nature of the methodology for development of the approach to the synthesis of hyperboloid gear drives is explained. In this regard, the mathematical model for synthesis of spatial transmissions and method for studying the conjugate active tooth surfaces with linear contact are presented. The methodology illustrated in this work is oriented to study the singularity of the active tooth surfaces of the gears. The results of the work include the theoretical and analytical definitions of the possible singular points types, which may occur in the contact area of the synthesized spatial transmissions. An algorithm for registration and elimination of the singularity of first order in the pole of meshing (pitch contact point) and its close vicinity is presented. The reason for the asymmetry of the crossed section of the teeth of the hyperboloid gear drives with face mating and the consequences of it for their exploitation are clarified.

**CONFLICT OF INTERESTS**

The authors have not declared any conflict of interests

**REFERENCES**


