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*Full Length Research Paper*

# Optimal homotopy perturbation method for solving partial differential equations with large solution domain

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**In this paper, an efficient modification of homotopy perturbation method, namely optimal homotopy perturbation method, is introduced for solving linear and nonlinear partial differential equations with large solution domain based on a new homotopy perturbation method and Padé approximation method. We compare the performance of the method with those of new homotopy perturbation and optimal variational iteration methods via three partial differential equations with large solution domain. Numerical results explicitly reveal that the suggested technique is highly capable to control the convergence region of approximate solution.**

**Key words:** New homotopy perturbation method, Pade´ approximation, optimal variational iteration method.

## INTRODUCTION

Approximate analytical schemes as the variational iteration method (VIM) (He, 1999) and homotopy perturbation method (HPM) (He, 1999) have been very widely used to solve partial differential equations (PDEs) for many applications in science and engineering (Abbasbandy, 2007; Biazar et al., 2007; Dehghan and Shakeri, 2008; Ganji, 2006, 2007; Jafari1 et al., 2011; Mohyud-Din and Noor, 2009; Mohyud-Din et al., 2009; Mohyud-Din, 2010, 2011; Hosseini et al., 2011; Hosseini et al., 2010; Yildirim, 2009; Abassy et al., 2007; Aminkhah and Biazar, 2010; He, 2004, 2005, 2006; He and Wu, 2007; Biazar and Eslami, 2011; He, 2006; Abdou and Soliman, 2005; Rashidi et al., 2011; Shahmohamadi and Rashidi, 2010; Rashidi et al., 2011). There are also many modifications of these introduced techniques, among which Aminkhah and Biazar's modification of HPM (a new homotopy perturbation method NHPM) (Aminkhah and Biazar, 2010) is much more attractive, where the homotopy perturbation method is coupled with the auxiliary parameters, and only one iteration leads to ideal results and (Hosseini et al., 2011) modification of VIM (optimal variational iteration method (OVIM) has also been caught much attention, where an auxiliary constant was introduced to adjust the control the convergence region of approximate solution.

In this paper, Aminkhah and Biazar's modification of HPM (Aminkhah and Biazar, 2010) is further extended, and a convenient way is suggested how to obtain suitable approximate solution in large solution domain. This method is called optimal homotopy perturbation method (OHPM) which is capable very effective in solving PDEs with large solution domain. Three examples are given to elucidate the performance of this method. Comparison with the results obtained by the NHPM (Aminkhah and Biazar, 2010) and OVIM (Hosseini et al., 2010) shows that the OHPM have remarkable accuracy. It is to be highlighted that Rashidi et al. (Rashidi et al., 2011; Shahmohamadi and Rashidi, 2010; Rashidi et al., 2011) presented some very useful and highly efficient modifications in variational iteration method. It is to be highlighted that Homotopy Perturbation Method (HPM) was formulated by taking full advantage of the standard homotopy and perturbation methods. The homotopy perturbation method (HPM) has been applied to a wide class of functional equations (He, 2004, 2005, 2006; Biazar et al., 2007; Dehghan and Shakeri, 2008; Ganji,

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2006, 2007; Jafari et al., 2011; Mohyud-Din and Noor, 2009; Mohyud-Din et al., 2009; Mohyud-Din, 2010, 2011; Hosseini et al., 2011; Hosseini et al., 2010; Yildirim, 2009; Abassy et al., 2007). It is worth mentioning that He (He, 2004) while comparing the homotopy analysis method (HAM) and homotopy perturbation method (HPM) clearly proved that HAM is a generalized Taylor series method which gives an infinite series solution and is coupled with all the deficiencies and limitations of this technique to have practical examples. Moreover, such schemes are not compatible to cope with the secular terms arising in the higher-order approximate solutions, whereas homotopy perturbation method (HPM) searches an asymptotic solution with few terms (mostly 2 to 4 terms) and does not require any convergence theory. The subsequent work (He, 2004, 2005, 2006) has explicitly strengthened this claim.

## METHODOLOGY

### Optimal variational iteration method

Consider the partial differential equations (PDEs),

$$\frac{\partial u}{\partial t} + N(x,t,u) = g(x,t), \quad (1)$$

With the following initial condition:

$$u(x,t_0) = f(x),$$

Where *N* is a nonlinear operator and g is inhomogeneous term. An unknown auxiliary parameter *h* can be inserted into the variational iteration algorithm. For solving Equation (1) by OVIM (Hosseini et al., 2010) we consider the following algorithm,

$$\begin{cases} u_0(x,t) \text{ is an orbitary function}, \\ u_1(x,t,h) = u_0(x,t) + h\int_{t_0}^{t} \lambda(s)\, N(x,s)\, ds \\ u_{n+1}(x,t,h) = u_n(x,t,h) + h\int_{t_0}^{t} \lambda(s) N(x,s,h)\, ds \qquad n \geq 1 \end{cases} \quad (2)$$

Where $\lambda$ is a Lagrange multiplier which can be identified optimally via variational theory (He, 2006). It should be emphasized that $u_n(x,t,h), n \geq 1$ can be computed by symbolic computation software such as Maple or Mathematica. The approximate solutions $u_n(x,t,h), n \geq 1$ contains the auxiliary parameter $h$ . The validity of the method is based on such an assumption that the approximation $u_n(x,t,h)$ converges to the exact solution. It is the auxiliary parameter $h$ that ensures that the assumption can be satisfied. In general, by means of the so-called $h$ -curve, it is straightforward to choose a proper value of $h$ which ensures that the approximate solutions are convergent (Hosseini et al., 2011). It has been shown that the OVIM is capable to approximate the solution more accurately than VIM in a large solution domain (Hosseini et al., 2010; Hosseini et al., 2011).

### Optimal homotopy perturbation method

For solving Equation (1) by the optimal homotopy perturbation method, at the first we solve this equation by NHPM (Aminkhah and Biazar, 2010) and for this reason we construct the following homotopy:

$$(1-p)\left(\frac{\partial U}{\partial t} - u_0\right) + p\left(\frac{\partial U}{\partial t} + N(x,t,U) - g(x,t)\right) = 0, \quad (3)$$

or equivalently,

$$\frac{\partial U}{\partial t} = u_0 - p(u_0 + N(x,t,U) - g(x,t)). \quad (4)$$

Applying the inverse operator, $L^{-1} = \int_{t_0}^{t}(.)dt,$ to both sides of Equation (4), we obtain:

$$U(x,t) = U(x,t_0) + \int_{t_0}^{t} u_0 dt - p\left(\int_{t_0}^{t} u_0 + N(x,t,U) - g(x,t)\right)dt, \quad (5)$$

Where $U(x,t_0) = u(x,t_0)$ is the initial condition for Equation (1).

Suppose the solution of Equation (5) has the following form:

$$U = U_0 + pU_1 + p^2 U_2 + \cdots, \quad (6)$$

Where $U_i\ i = 0,1,2,\ldots$ are functions which should be determined. Now suppose that the initial approximation of the solutions $u_0$ has the form:

$$u_0(x,t) = \sum_{i=0}^{N} \alpha_i(x) P_i(t), \quad (7)$$

Where $\alpha_i(x)$ are unknown coefficients and $P_0(t),\ P_1(t), P_2(t),\ldots$ are specific functions. Substituting (6) into (5) and equating the coefficients of *p* with same power leads to

$$\begin{aligned} &p^0: U_0(x,t) = f(x) + \textstyle\sum_{i=0}^{N} \alpha_i(x)\int_{t_0}^{t} P_i(t)dt, \\ &p^1: U_1(x,t) = -\textstyle\sum_{i=0}^{N} \alpha_i(x)\int_{t_0}^{t} P_i(t)dt - \int_{t_0}^{t}(N(x,t,U_0) - g(x,t))dt, \\ &p^2: U_2(x,t) = -\int_{t_0}^{t} N(x,t,U_0,U_1)dt, \\ &p^3: U_3(x,t) = -\int_{t_0}^{t} N(x,t,U_0,U_1,U_2)dt, \\ &\vdots \\ &p^j: U_j(x,t) = -\int_{t_0}^{t} N\left(x,t,U_0,U_1,U_2,\ldots,U_{j-1}\right)dt, \end{aligned} \quad (8)$$

Now if we solve the equation in such a way that $U_1(x,t)$=0, then Equation (8) result in $U_2(x,t) = U_3(x,t) = \cdots = 0$ ; therefore, the solution can be obtained by using

$$u(x,t) = U_0(x,t) = f(x) + \sum_{i=0}^{N} \alpha_i(x)\int_{t_0}^{t} P_i(t)dt. \quad (9)$$

It is worthwhile to note that if $U_1(x,t)$ be analytic at $t = t_0$, then its Taylor series:

**Table 1.** Comparison maximum absolute error for Example1.

| N | (OHPM) | Time (S) | NHPM | Time (S) | OVIM | *Optimal h* | Time (S) |
|---|---|---|---|---|---|---|---|
| 10 | $3\times 10^{-1}$ | 0.65 | $2.2\times 10^{8}$ | 0.26 | $4.4\times 10^{-1}$ | -0.07 | 0.73 |
| 20 | $1.3\times 10^{-2}$ | 0.95 | $2.8\times 10^{13}$ | 0.63 | $2.7\times 10^{-1}$ | -0.13 | 2.65 |
| 30 | $8.8\times 10^{-5}$ | 1.26 | $2.2\times 10^{16}$ | 0.93 | $3.6\times 10^{-2}$ | -0.25 | 6.24 |
| 40 | $1.1\times 10^{-7}$ | 1.98 | $6.2\times 10^{19}$ | 1.22 | $8.5\times 10^{-4}$ | -0.36 | 10.81 |
| 50 | $3\times 10^{-11}$ | 2.56 | $1.4\times 10^{16}$ | 1.88 | $9.5\times 10^{-7}$ | -0.54 | 18.65 |
| 60 | $1.9\times 10^{-15}$ | 3.41 | $4.8\times 10^{19}$ | 2.13 | $3.3\times 10^{-10}$ | -0.59 | 29.39 |
| 70 | $3.6\times 10^{-20}$ | 4.12 | $2.9\times 10^{16}$ | 2.65 | $8.8\times 10^{-15}$ | -0.61 | 53.30 |
| 80 | $2.1\times 10^{-25}$ | 5.09 | $4.4\times 10^{16}$ | 3.23 | $7.9\times 10^{-19}$ | -0.67 | 93.16 |
| 90 | $4.2\times 10^{-31}$ | 6.35 | $1.9\times 10^{12}$ | 4.01 | $5.6\times 10^{-25}$ | -0.71 | 109.23 |
| 100 | $3.3\times 10^{-37}$ | 7.68 | $2.8\times 10^{9}$ | 4.76 | $3.4\times 10^{-31}$ | -0.75 | 126.36 |

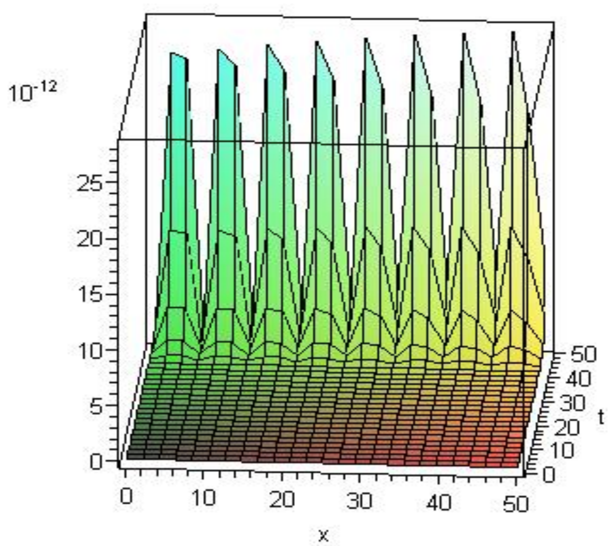


**Figure 1.** Absolute error of OHPM with *N=50, L=M=25*, for example 2.

$U_1(x,t) = \sum_{n=0}^{N} a_n(x)(t-t_0)^n$ can be used in Equation(8) where $a_n(x)$ are known coefficients.

Now we apply *t-pade* approximation technique on the obtained solution (9). Note that t-*Pade* approximation $\left[\frac{L}{M}\right] = \frac{P_L(x,t)}{Q_M(x,t)}$ is a rational approximation to $u(x,t)$ where $P_L(x,t)$ is a polynomial of *t* of degree at most L and $Q_M(x,t)$ is a polynomial of *t* of degree at most M such as:

$$u(x,t) - \frac{P_L(x,t)}{Q_M(x,t)} = O(t^{L+M+1}), \qquad (10)$$

For a fixed value of $L+M$ the error is smallest when $P_L(x,t)$ and $Q_M(x,t)$ have the same degree or when $P_L(x,t)$ has degree one higher than $Q_M(x,t)$, with respect to *t*. In this paper we assume that *N* is even and $L = M = \frac{N}{2}$. In fact, the proposed technology is very simple, easier to implement and is capable to approximate the solution more accurately in a large solution domain.

## RESULTS

To demonstrate the effectiveness of the OHPM and to compare this method with the NHPM (Aminkhah and Biazar, 2010), and OVIM (Hosseini et al., 2010; Hosseini et al., 2011), we considered three PDEs with large solution domain. The computations associated the examples are performed using Maple 12.

### Example 1

Consider the following evolution equation:

$$\begin{cases} u_t + u_x = 2u_{xt} & t > 0, \quad 0 \le x \le 50 \\ u(x,0) = e^{-x} & 0 \le x \le 50 \end{cases} \qquad (11)$$

with the exact solution $u(x,t) = e^{-x-t}$.

We take the solution domain as $(x,t) \in [0,50] \times [0,50]$, the absolute error of OHPM, NHPM and OVIM and their computational times are given with different N in Table 1.

### Example 2

Consider the following evolution equation:

$$\begin{cases} u_t + u_{xxxx} = 0, & t > 0, \quad 0 \le x \le 50 \\ u(x,0) = \sin(x), & 0 \le x \le 50 \end{cases} \qquad (12)$$

with the exact solution $u(x,t) = e^{-t}\sin(x)$.

Figures 1, 2 and 3 show the absolute error of $u_{50}(x,t)$

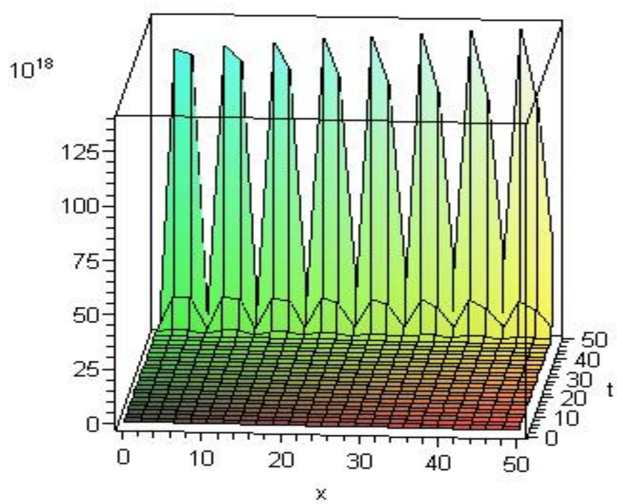


**Figure 2.** Absolute error of NHPM with *N=50*, for example 2.

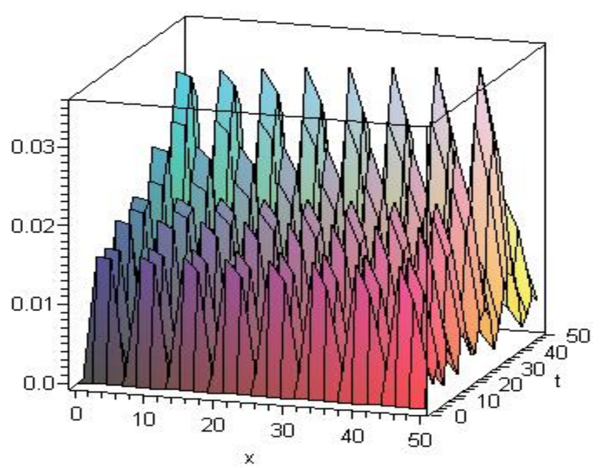


**Figure 3.** Absolute error of OVIM with N=50, h=0.05 for example 2.

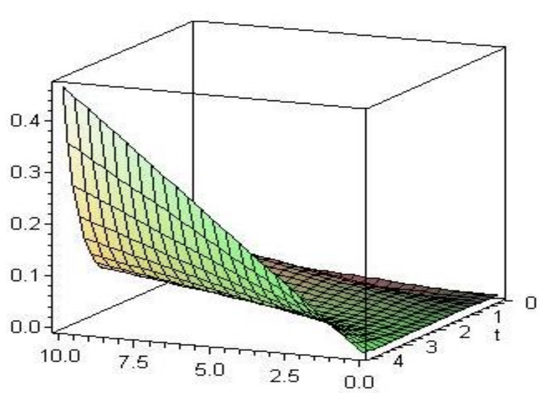


**Figure 4.** Absolute error of OVIM with *N=5, h=0.43* for example 3.

on $(x,t) \in [0,50] \times [0,50]$, for OHPM, NHPM and OVIM, respectively.

### Example 3

Consider the following evolution equation:

$$\begin{cases} u_t - u_{xxt} + uu_x = 0 \ , & t > 0, \quad 0 \leq x \leq 10 \\ u(x,0) = x \ , & 0 \leq x \leq 10 \end{cases} \quad (13)$$

with the exact solution $u(x,t) = \frac{x}{1+t}$.

Solving this problem via NHPM, we find $u_2(x,t) = x - xt + xt^2$, and then applying OHPM with *L=M=1*, we have $u_2(x,t) = \frac{x}{1+t}$ which is the exact solution. Also the absolute error of OVIM for $u_5(x,t)$ on $(x,t) \in [0,10] \times [0,4]$ is shown in Figure 4.

## Conclusion

The present technology provides a simple way to adjust and control the convergence region of approximate solution for any values of *t* and *x.* Numerical results explicitly reveal the complete reliability, efficiency and accuracy of the suggested technique. It needs to be highlighted that the optimal homotopy perturbation algorithm is capable to reduce computational time and obtaining suitable approximate solution of PDEs in large solution domain.


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