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Homotopy analysis method for Zakharov-Kuznetsov (ZK) equation with fully nonlinear dispersion

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In this paper, we apply Homotopy Analysis Method (MHAM) to find appropriate solutions of Zakharov-Kuznetsov (ZK) equations. Numerical results coupled with graphical representation explicitly reveal the complete reliability of the proposed algorithm.

Key words: Homotopy analysis method, Zakharov-Kuznetsov (ZK) equation, exact solutions, MAPLE.

INTRODUCTION

The rapid development of nonlinear sciences (Abbasbandy, 2008; Abbasbandy and Zakaria, 2008; Abbasbandy, 2006, 2007, 2010, 2011; Liao, 2003, 2004, 2005; Gurtin and Maccamy, 1977; Gurney and Nisbet, 1975; Lu, 2000; Tan and Abbasbandy, 2008; Hayat et al., 2004; Hayat and Khan, 2005; Mohyud-Din and Yildirim, 2010; Yildirim and Mohyud-Din, 2010) witnesses a wide range of analytical and numerical techniques by various scientists. Most of the developed schemes have their limitations like limited convergence, divergent results, linearization, discretization, unrealistic assumptions and non-compatibility with the versatility of physical problems (Abbasbandy, 2008; Abbasbandy and Zakaria, 2008; Abbasbandy, 2006, 2007, 2010, 2011; Liao, 2003, 2004, 2005; Gurtin and Maccamy, 1977; Gurney and Nisbet, 1975). In a similar context, Liao (2003, 2004, 2005) developed Homotopy Analysis Method (HAM) which is being applied on a wide range of nonlinear problems of physical nature (Abbasbandy, 2008; Abbasbandy and Zakaria, 2008; Abbasbandy, 2006, 2007, 2010, 2011; Gurtin and Maccamy, 1977; Gurney and Nisbet, 1975; Lu, 2000; Tan and Abbasbandy, 2008; Hayat et al., 2004; Hayat and Khan, 2005; Mohyud-Din and Yildirim, 2010; Yildirim and Mohyud-Din, 2010) and the references therein. The basic motivation of the present study is the extension of a very reliable and efficient technique which is called the Homotopy Analysis Method (HAM) to tackle Zakharov-Kuznetsov (ZK) equations. Liu (2010a, b, 2011) applied this technique for different values of h and obtained the efficient results. It is observed that the proposed algorithm is fully compatible with the complexity of such equations. Numerical examples are given which reveal the efficiency and reliability of the proposed algorithm (HAM).

ANALYSIS OF HOMOTOPY ANALYSIS METHOD (HAM)

Consider the following differential equation (Jafari and Seifi, 2009)

$$N[u(\tau,t)] = 0, \qquad (1)$$

where N is a nonlinear operator, τ and t

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denotes independent variables, $u(\tau,t)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in a similar way. By means of generalizing the traditional homotopy method, Liao constructs the so called zero - order deformation equation.

$$(1-p)L[\varphi(\tau,t;p)-u_0(\tau,t)] = phH(\tau,t)N[\varphi(\tau,t;p)] \quad (2)$$

where $p\in[0,1]$ is the embedding parameter, $h\neq 0$ is a nonzero parameter, $H(\tau,t)\neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(\tau,t)$ is an initial guess of $u(\tau,t)$, $u(\tau,t;p)$ is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when and, $p=0$ and $p=1$ it holds

$$\varphi(\tau,t;0) = u_0(\tau,t),$$

$$\varphi(\tau,t;1) = u(\tau,t),$$

respectively. Thus, as p increases from 0 to 1, the solution $\varphi(\tau,t;p)$ varies from the initial guesses $u_0(\tau,t)$ to the solution $u(\tau,t)$. Expanding in Taylor series with respect to p, we have

$$\varphi(\tau,t;p) = u_0(\tau,t) + \sum_{m=1}^{\infty} u_m(\tau,t)p^m,$$

where $u_m(\tau,t) = \frac{1}{m!}\frac{\partial^m \varphi(\tau,t;p)}{\partial p^m}$ at $p=0.$

If the auxiliary linear operator, the initial guess, the auxiliary h, and the auxiliary function are so properly chosen, the above series converges at $p=1$; then we have

$$u(\tau,t) = u_0(\tau,t) + \sum_{m=1}^{\infty} u_m(\tau,t),$$

We define the vector

$$\overrightarrow{u_n} = \{u_0(\tau,t), u_1(\tau,t), u_2(\tau,t), \dots u_n(\tau,t)\}$$

Differentiating Equation (2) m times with respect to the embedding parameter p and then setting $p=0$ and finally dividing them by $m!$, we obtain the mth-order deformation equation

$$L[u_m(\tau,t) - \chi_m u_{m-1}(\tau,t)] = hH(\tau,t)R_m(\vec{u}_{m-1},\tau,t), \quad (3)$$

where $R_m(\vec{u}_{m-1},\tau,t) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\varphi(\tau,t;p)]}{\partial p^{m-1}}$ at $p=0.$

and

$$\chi_m = \begin{cases} 0 & m\leq 1, \\ 1 & m>1. \end{cases}$$

Applying L^{-1} to both sides of (3), we get

$$u_m(\tau,t) = \chi_m u_{m-1}(\tau,t) + hL^{-1}[H(\tau,t)R_m(\vec{u}_{m-1},\tau,t)]$$

In this way, it is easy to obtain u_n for m $\geq$ 1; at mth-order, we have

$$u(\tau,t) = \sum_{m=0}^{M} u_m(\tau,t).$$

when $M\to\infty$, we get an accurate approximation of the original Equation (1). For convergence of the above method, see Liao's researches. If Equation (1) admits unique solution, then this method will produce the unique solution. If Equation (1) does not possess unique solution, HAM will give a solution among many other (possible) solutions.

NUMERICAL APPLICATIONS

Here, we apply homotopy analysis method (HAM) for finding appropriate solutions of Zakharov-Kuznetsov (ZK) equations. Numerical results are very encouraging.

Example 1

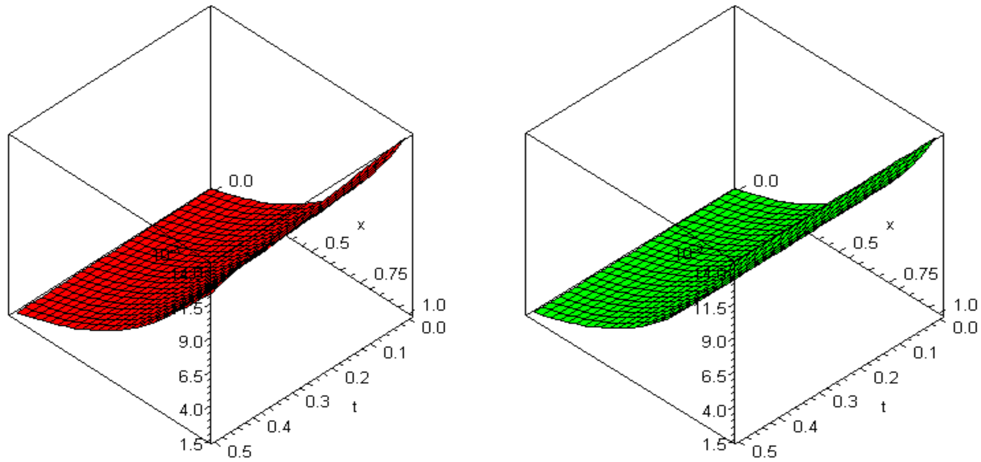
We first consider the following $ZK(2,2,2)$ equation.

$$u_t + (u^2)_x + \frac{1}{8}(u^2)_{xxx} + \frac{1}{8}(u^2)_{yyx} = 0,$$

$$u(x,y,0) = \frac{4}{3}\lambda\sinh^2(x+y).$$

To solve the given Equation by HAM, we choose the linear operator

$$L[\varphi(x,y,t;q)] = \frac{\partial}{\partial t}\left(\varphi(x,y,t;q)\right),$$



(a) (b)

Figure 1(a and b). Represent the graphical representation of exact and approximate solution in the domain $t \in (0,0.5)$ and $x \in (0,1)$ when $y = 0.9$ and $\lambda = 0.001$.

with the property

$$L[c_1] = 0,$$

where c_1 is the integral constants. The inverse operator L^{-1} is given by

$$L^{-1} = \int_0^t (.)dt,$$

Now, we define a non linear operator as

$$N[\varphi(x,y,t;q)] =$$
$$\varphi(x,y,t;q)_t + (\varphi(x,y,t;q)^2)_x + \frac{1}{8}(\varphi(x,y,t;q)^2)_{xxx} + \frac{1}{8}(\varphi(x,y,t;q)^2)_{yyx}$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-q)L[\varphi(x,y,t;q) - u_0(x,y,t)] = qhH(x,y,t)N[\varphi(x,y,t;q)].$$

for $q = 0$ and $q = 1$, we can write

$$\varphi(x,y,t;0) = u_0(x,y,t), \qquad \varphi(x,y,t;1) = u(x,y,t).$$

Thus, we obtain the mth $-$order deformation equation

$$L[u_m(x,y,t) - \chi_m u_{m-1}(x,y,t)] = hH(x,y,t)R_m(\vec{u}_{m-1}).$$

with initial condition $u_m(x,y,0) = 0$,

where

$$R_m(\vec{u}_{m-1}) = h\left[(u_{m-1}(x,y,t))_t + (\sum_{r=0}^{m-1} u_r(x,y,t)\,u_{m-1-r}(x,y,t))_x + \frac{1}{8}\left(\sum_{r=0}^{m-1} u_r(x,y,t)\,u_{m-1-r}(x,y,t)\right)_{xxx} + \frac{1}{8}\left(\sum_{r=0}^{m-1} u_r(x,y,t)\,u_{m-1-r}(x,y,t)\right)_{yyx}\right].$$

Now the solutions of the mth $-$order deformation equation are

$$u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + L^{-1}[hH(x,y,t)R_m(\vec{u}_{m-1})], m \geq 1,$$

we start with an initial approximation

$$u_0(x,y) = \frac{4}{3}\lambda \sinh^2(x+y),$$

by means of the iteration formula as discussed above, if $h = -1, H = 1$, we can obtain directly the other components (Figure 1) as

$$u_1 = -\frac{32}{9}\sinh(x+y)\cosh(x+y)\,(10\cosh^2(x+y) - 7)\lambda^2 t,$$

$$u_2 = \frac{64}{27}\lambda^3 t^2(1200\cosh^6(x+y) - 2080\cosh^4(x+y) + 968\cosh^2(x+y) - 79),$$

$$u_3 = -\frac{4096}{243}\lambda^4 t^3 \sinh(x+y)\cosh(x+y)(22665\cosh^2(x+y) - 42900\cosh^4(x+y) + 23800\cosh^6(x+y) - 3142)$$

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$$u(x,y,t)=\frac{4}{3}\lambda\sinh^2(x+y)-\frac{32}{9}\sinh(x+y)\cosh(x+y)\,(10\cosh^2(x+y)-7)\lambda^2t+\cdots$$

The rest of the components of the iteration formulae can be obtained using the MAPLE.

Example 2

Consider the following $ZK(2,2,2)$ equation.

$$u_t+(u^2)_x+\frac{1}{8}(u^2)_{xxx}+\frac{1}{8}(u^2)_{yyx}=0,$$

$$u(x,y,0)=-\frac{4}{3}\lambda\cosh^2(x+y).$$

To solve the given Equation by HAM we choose the linear operator

$$L[\varphi(x,y,t;q)]=\frac{\partial}{\partial t}(\varphi(x,y,t;q)),$$

with the property

$$L[c_1]=0,$$

where c_1 is the integral constants. The inverse operator L^{-1} is given by

$$L^{-1}=\int_0^t(.)dt,$$

we now define a nonlinear operator as

$$N[\varphi(x,y,t;q)]=\varphi(x,y,t;q)_t+(\varphi(x,y,t;q)^2)_x+\frac{1}{8}(\varphi(x,y,t;q)^2)_{xxx}+\frac{1}{8}(\varphi(x,y,t;q)^2)_{yyx}$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-q)L[\varphi(x,y,t;q)-u_0(x,y,t)]=qhH(x,y,t)N[\varphi(x,y,t;q)].$$

For $q=0$ and $q=1$, we can write

$$\varphi(x,y,t;0)=u_0(x,y,t),\qquad \varphi(x,y,t;1)=u(x,y,t).$$

Thus, we obtain the mth $-$order deformation equation

$$L[u_m(x,y,t)-\chi_m u_{m-1}(x,y,t)]=hH(x,y,t)R_m(\vec{u}_{m-1}).$$

with initial condition $u_m(x,y,0)=0,$

where

$$R_m(\vec{u}_{m-1})=h\left[(u_{m-1}(x,y,t))_t+(\sum_{r=0}^{m-1}u_r(x,y,t)u_{m-1-r}(x,y,t))_x+\frac{1}{8}\left(\sum_{r=0}^{m-1}u_r(x,y,t)u_{m-1-r}(x,y,t)\right)_{xxx}+\frac{1}{8}\left(\sum_{r=0}^{m-1}u_r(x,y,t)u_{m-1-r}(x,y,t)\right)_{yyx}\right]$$

Now the solutions of the mth $-$order deformation equation are

$$u_m(x,y,t)=\chi_m u_{m-1}(x,y,t)+L^{-1}[hH(x,y,t)R_m(\vec{u}_{m-1})],m\geq 1,$$

We start with an initial approximation

$$u_0(x,y)=-\frac{4}{3}\lambda\cosh^2(x+y),$$

by means of the iteration formula as discussed above; if $h=-1,H=1,$ we can obtain directly the others components (Figure 2) as

$$u_1=-\frac{32}{9}\sinh(x+y)\cosh(x+y)\,(10\cosh^2(x+y)-3)\lambda^2t,$$

$$u_2=-\frac{64}{27}\lambda^3t^2(1200\cosh^6(x+y)-1520\cosh^4(x+y)+408\cosh^2(x+y)-9),$$

$$u_3=-\frac{4096}{243}\lambda^4t^3\sinh(x+y)\cosh(x+y)(8265\cosh^2(x+y)-28500\cosh^4(x+y)+23800\cosh^6(x+y)-423)$$

$$u(x,y,t)=-\frac{4}{3}\lambda\cosh^2(x+y)-\frac{32}{9}\sinh(x+y)\cosh(x+y)\,(10\cosh^2(x+y)-3)\lambda^2t+\cdots$$

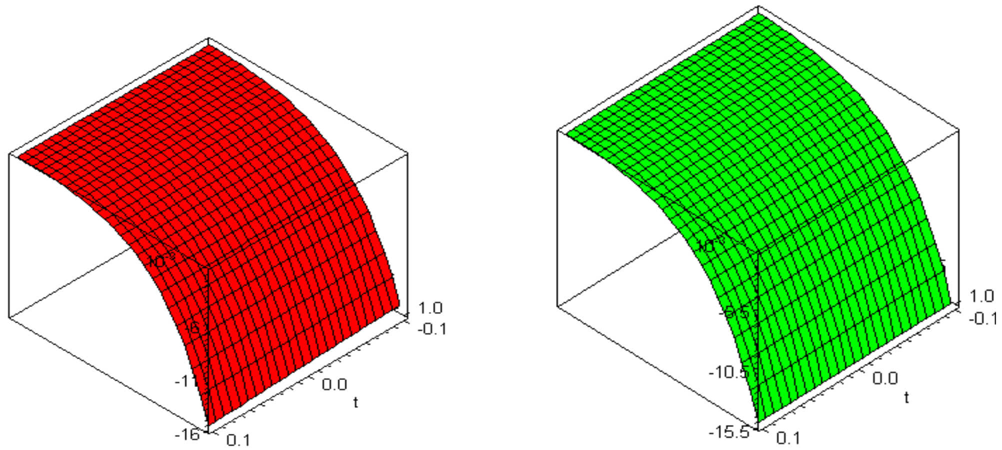
Rest of the components of the iteration formulae can be obtained using the Maple.

Example 3

Consider the following $ZK(3,3,3)$ equation.

$$u_t+(u^3)_x+2(u^3)_{xxx}+2(u^3)_{yyx}=0,$$

$$u(x,y,0)=\frac{3}{2}\lambda\sinh\left(\frac{1}{6}(x+y)\right).$$



(a) (b)

Figure 2(a and b). Representation of the graphical representation of exact and approximate solution in the domain $t \in (-0.1, 0.1)$ and $x \in (-1, 1)$ when $y = 0.9$ and $\lambda = 0.001$.

To solve the given Equation by HAM we choose the linear operator

$$L[\varphi(x,y,t;q)] = \frac{\partial}{\partial t}\left(\varphi(x,y,t;q)\right),$$

with the property

$$L[c_1] = 0,$$

where c_1 is the integral constants.

The inverse operator L^{-1} is given by

$$L^{-1} = \int_0^t (.)dt,$$

We now define a nonlinear operator as

$$N[\varphi(x,y,t;q)] = \varphi(x,y,t;q)_t + (\varphi(x,y,t;q)^3)_x + 2(\varphi(x,y,t;q)^3)_{xxx} + 2(\varphi(x,y,t;q)^3)_{yyx}$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-q)L[\varphi(x,y,t;q) - u_0(x,y,t)] = qhH(x,y,t)N[\varphi(x,y,t;q)].$$

for $q = 0$ and $q = 1$, we can write

$$\varphi(x,y,t;0) = u_0(x,y,t), \qquad \varphi(x,y,t;1) = u(x,y,t).$$

Thus, we obtain the mth $-$order deformation equation.

$$L[u_m(x,y,t) - \chi_m u_{m-1}(x,y,t)] = hH(x,y,t)R_m(\vec{u}_{m-1}).$$

with initial condition $u_m(x,y,0) = 0,$

where

$$R_m(\vec{u}_{m-1}) = h\Big[\big(u_{m-1}(x,y,t)\big)_t + \big(\textstyle\sum_{r=0}^{m-1}\sum_{k=0}^{s} u_s(x,y,t)\,u_{s-k}(x,y,t)\,u_{m-1-r}(x,y,t)\big)_x + 2\big(\sum_{r=0}^{m-1}\sum_{k=0}^{s} u_s(x,y,t)\,u_{s-k}(x,y,t)\,u_{m-1-r}(x,y,t)\big)_{xxx} + 2\big(\sum_{r=0}^{m-1}\sum_{k=0}^{s} u_s(x,y,t)\,u_{s-k}(x,y,t)\,u_{m-1-r}(x,y,t)\big)_{yyx}\Big].$$

Now the solutions of the mth $-$order deformation equation are

$$u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + L^{-1}[hH(x,y,t)R_m(\vec{u}_{m-1})], m \geq 1,$$

We start with an initial approximation

$$u_0(x,y) = \frac{3}{2}\lambda \sinh\left(\frac{1}{6}(x+y)\right)$$

By means of the iteration formula as discussed above, if $h = -1, H = 1,$ we can obtain directly the other components (Figure 3) as

$$u_1 = -\frac{3}{8}\cosh\left(\frac{1}{6}(x+y)\right)\left(9\cosh^2\left(\frac{1}{6}(x+y)\right) - 8\right)\lambda^3 t,$$

$$u_2 = \frac{3}{64}\lambda^5 t^2 \sinh\left(\frac{1}{6}(x+y)\right)\left(765\cosh^4\left(\frac{1}{6}(x+y)\right) - 729\cosh^2\left(\frac{1}{6}(x+y)\right) + 91\right),$$

$$u_3 = -\frac{1}{256}\lambda^7 t^3 \cosh\left(\frac{1}{6}(x+y)\right)\left(-382293\cosh^4\left(\frac{1}{6}(x+y)\right) + 234468\cosh^2\left(\frac{1}{6}(x+y)\right) + 188181\cosh^6\left(\frac{1}{6}(x+y)\right) - 39851\right)$$

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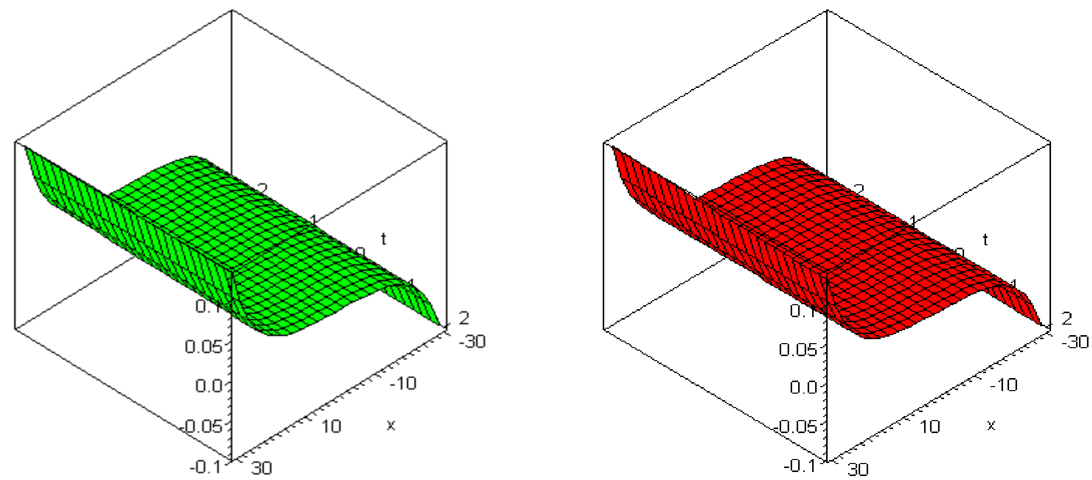


(a) (b)

Figure 3(a and b). Representation of the graphical representation of exact and approximate solution in the domain $t \epsilon (-30,30)$ and $x \epsilon (-2,2)$ when $y = 0.9$ and $\lambda = 0.001$.

$$u(x,y,t) = \frac{3}{2}\lambda \sinh\left(\frac{1}{6}(x+y)\right) - \frac{3}{8}\cosh\left(\frac{1}{6}(x+y)\right)\left(9\cosh^2\left(\frac{1}{6}(x+y)\right) - 8\right)\lambda^3 t + \cdots$$

The rest of the components of the iteration formulae can be obtained using the MAPLE.

Example 4

We first consider the following $ZK(3,3,3)$ equation.

$$u_t + (u^3)_x + \frac{1}{8}(u^3)_{xxx} + \frac{1}{8}(u^3)_{yyx} = 0,$$

$$u(x,y,0) = \frac{3}{2}\lambda \cosh\left(\frac{1}{6}(x+y)\right).$$

To solve the given Equation by HAM we choose the linear operator

$$L[\varphi(x,y,t;q)] = \frac{\partial}{\partial t}(\varphi(x,y,t;q)),$$

with the property

$$L[c_1] = 0,$$

where c_1 is integral constants.

The inverse operator L^{-1} is given by

$$L^{-1} = \int_0^t (.)dt,$$

We now define a nonlinear operator as

$$N[\varphi(x,y,t;q)] = \varphi(x,y,t;q)_t + (\varphi(x,y,t;q)^3)_x + \frac{1}{8}(\varphi(x,y,t;q)^3)_{xxx} + \frac{1}{8}(\varphi(x,y,t;q)^3)_{yyx}$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-q)L[\varphi(x,y,t;q) - u_0(x,y,t)] = qhH(x,y,t)N[\varphi(x,y,t;q)].$$

for $q = 0$ and $q = 1$, we can write

$$\varphi(x,y,t;0) = u_0(x,y,t), \qquad \varphi(x,y,t;1) = u(x,y,t).$$

Thus, we obtain the mth $-$order deformation equation.

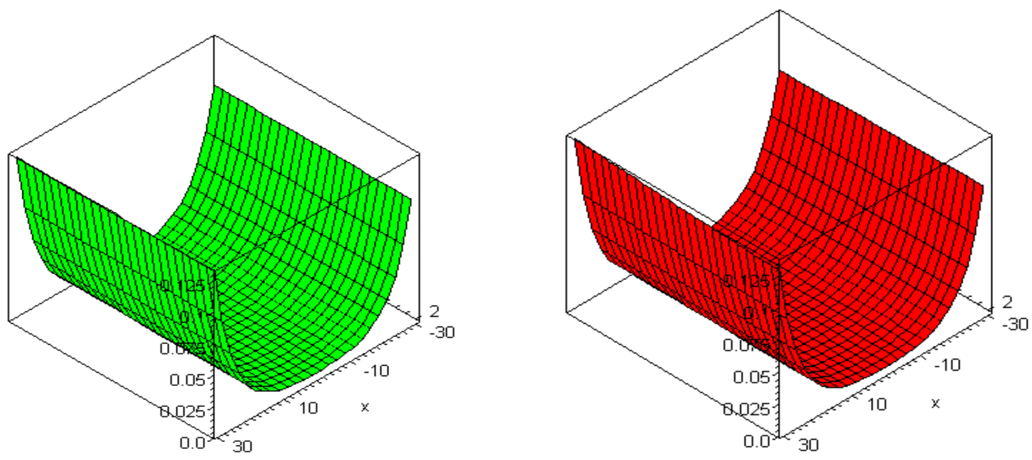
$$L[u_m(x,y,t) - \chi_m u_{m-1}(x,y,t)] = hH(x,y,t)R_m(\vec{u}_{m-1}).$$

with initial condition $u_m(x,y,0) = 0,$

where

$$R_m(\vec{u}_{m-1}) = h\Big[\left(u_{m-1}(x,y,t)\right)_t + \left(\sum_{r=0}^{m-1}\sum_{k=0}^{s} u_s(x,y,t)\,u_{s-k}(x,y,t)\,u_{m-1-r}(x,y,t)\right)_x + \frac{1}{8}\left(\sum_{r=0}^{m-1}\sum_{k=0}^{s} u_s(x,y,t)\,u_{s-k}(x,y,t)\,u_{m-1-r}(x,y,t)\right)_{xxx} + \frac{1}{8}\left(\sum_{r=0}^{m-1}\sum_{k=0}^{s} u_s(x,y,t)\,u_{s-k}(x,y,t)\,u_{m-1-r}(x,y,t)\right)_{yyx}\Big].$$

Now the solutions of the mth $-$order deformation equation are



(a) (b)

Figure 4(a and b). Representation of the graphical representation of exact and approximate solution in the domain $t \in (-30,30)$ and $x \in (-2,2)$ when $y = 0.9$ and $\lambda = 0.001$.

$$u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + L^{-1}[hH(x,y,t)R_m(\vec{u}_{m-1})], m \geq 1,$$

We start with an initial approximation

$$u_0(x,y) = \frac{3}{2}\lambda\cosh\left(\frac{1}{6}(x+y)\right),$$

By means of the iteration formula as discussed above, if $h = -1, H = 1,$ we can obtain directly the other components (Figure 4) as

$$u_1 = -\frac{3}{8}\sinh\left(\frac{1}{6}(x+y)\right)\left(9\cosh^2\left(\frac{1}{6}(x+y)\right) - 1\right)\lambda^3 t,$$

$$u_2 = \frac{3}{64}\lambda^5 t^2 \cosh\left(\frac{1}{6}(x+y)\right)\left(765\cosh^4\left(\frac{1}{6}(x+y)\right) - 801\cosh^2\left(\frac{1}{6}(x+y)\right) + 127\right),$$

$$u_3 = -\frac{1}{256}\lambda^7 t^3 \sinh\left(\frac{1}{6}(x+y)\right)\left(-182250\cosh^4\left(\frac{1}{6}(x+y)\right) + 34425\cosh^2\left(\frac{1}{6}(x+y)\right) + 188181\cosh^6\left(\frac{1}{6}(x+y)\right) - 505\right)$$

$$u(x,y,t) = \frac{3}{2}\lambda\cosh\left(\frac{1}{6}(x+y)\right) - \frac{3}{8}\sinh\left(\frac{1}{6}(x+y)\right)\left(9\cosh^2\left(\frac{1}{6}(x+y)\right) - 1\right)\lambda^3 t + \cdots$$

The rest of the components of the iteration formulae can be obtained using the MAPLE.

Conclusions

Homotopy Analysis Method (HAM) is applied to find appropriate solutions of Zakharov-Kuznetsov (ZK) equations. The proposed algorithm is fully capable to cope with the nonlinearity and complexity of the Zakharov-Kuznetsov (ZK) equations. Numerical results re-confirm the efficiency of the suggested scheme (HAM).

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