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Full Length Research Paper

Towards acceleration of Rump's fast and parallel circular interval arithmetic for enclosing solution of non linear system of equations

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It is often the practice to express measurements of experimental problems in terms of uncertainties either due to contamination of measuring instruments or inaccurate measurements in the experimental models. In this note as an attempt at solving a system of nonlinear equation by accelerating the convergence of Rump's fast and parallel interval arithmetic incorporating where in, the Carstensen and Petkovic circular arithmetic for enlarging a disk to be inverted in the complex plane. The problem of excess widths in the midpoint-radius matrix and midpoint –radius vector multiplication is taken into account by using the procedure of Ceberio and Kreinovich for fast multiplication of two interval matrices (or interval matrix and interval vector) whose entries are expressed in terms of midpoint-radius matrix. We used Interval Gaussian Elimination algorithm and Interval Gauss-Siedel iterative method as our basic tools with Newtonian steps, some significant gains over that of Rump's method were achieved. A stopping criterion for a Newton's step is given in terms of defect measurement instead of the error. AMS SUBJECT CLASSIFICATION (2000): 65 G20, 65G30, 65G40.

Key words: Rump's interval operation, zeros of nonlinear system of equation, Carstensen and Petkovic circular interval arithmetic for disk inversion.

INTRODUCTION

We consider the problem of solving a nonlinear system of equation.

$$F(x) = 0 \tag{1.1}$$

Where; $F : D \subset IR^n \to IR^n$, and we assume that F is a continuously differentiable uniformly monotone gradient map. We denote IR^n as interval vector and IR^{nxn} as interval matrix. We will expect the readers to be equipped with the basics of interval arithmetic. For introductory remarks on interval arithmetic see for example, Kearfott (1996); Hargreaves (2002); Alefeld and Mayer (2000) and also in section 2 of the paper. For $a, b \in IR$, the quantity $d(a,b) = Max\left\{\left|\underline{a}, \underline{b}\right|, \left|\overline{a}, \overline{b}\right|\right\} \subset IR$ is a metric, the Hausdorff distance. IR is complete with respect to this metric and we will always believe that IR^n is endowed with the corresponding product topology. In order to solve problem (1.1), we start from Taylor expansion of F:

$$0 = \mathrm{F}(\mathrm{x}^*) = F(x^{(o)}) + F'(x^{(o)})\,(x^* - x^{(o)}) + 0(//\,x^* - x^{(o)}\,//) \tag{1.2}$$

One then transforms the Non Linear interval System (1.1) via (1.2), to interval linear system.

$$F'(\hat{x}^{(k)} - x^{(k)}) \ni -F(x^{(k)}), \tag{1.3}$$

Where $F'(x^{(k)})$ is a suitable interval extension of the Jacobian matrix.

Over the box $x^{(k)}$ *and* $x \in x^{(o)}$, signifies a predictor or initial guess point.

If one assumes that the function F is an M – function, one then iterates $x^{(k+1)}$ with the Newtonian step.

$$x^{(k+1)} = x^{(k)} \bigcap \hat{x}^{(k)} \tag{1.4}$$

to provide tight inclusion bound for the sought zeros of F.

Bisection method may be employed to narrow the box if the coordinate intervals are not smaller than those of $X^{(k)}$, Kearfott (1996 and 1998).

The existence of zeros in a box can also be proved using any of the Kantorovich, Miranda and Borsuk's theorems, see for instance Alefeld et al. (2004). Kantorovich's theorem has always been motivated by the analysis of Newton's iteration to approximate the zeros of nonlinear system (1.1). This gives a priority criterion for the convergence of Newton's iteration which proves that there is a zero of F within a certain ball centered at the initial guess for Newton's method. Its basic ingredient in its formulation is the large Lipschtz- continuity of the derivative of F in a sufficiently large neighborhood of the starting point and that it is assumed that the function values at the starting point are small enough.

The layouts of the paper are as follows: In section 2 we give a brief review of interval arithmetic operations. In section 3, we described Rump's interval mid point matrix - radius operation. An improvement of this is given by enlarging a disk to be inverted using the ideas described in Carstensen and Petkovic (1994) aided by the approach of Ceberio and Kreinovich (2004) to accelerate the basic iterative methods. In this way better results are obtained. In section 4, the type of interval linear solvers adopted to achieve our results are described. In the case of interval Jacobian being singular, we described box splitting as a way of eliminating this problem. In section 4, numerical example is given and concluding remarks are given on the basis of these results. It is shown that our presented methods have substantial gains over previously known Rump's interval matrix operations.

The interval arithmetic

Interval arithmetic will be represented by boldface with brackets '' [ ]'' signifying the interval defined by an upper bound and a lower bound. Under scores will be used for lower bounds of intervals and over scores will be used for upper bounds. Similarly, for interval defined by a mid point and a radius the brackets "< >" will be used. A real interval is a set of the form;

x = $[\underline{x}, \overline{x}] = \{\underline{x} \le x \le \overline{x}\}$,

Where $\underline{x}$ is called the infimum and $\overline{x}$ is called the supremum. Thus the set of all intervals over R is denoted IR. Thus if x is a more complex expression, we also write;

$\underline{x}$ = inf (x), $\overline{x}$ = sup(x).

When $\underline{x} = \overline{x}$, we say that the interval x is a thin interval.

The interval x is called thick if $\underline{x} < \overline{x}$,

The Mid point of x, Mid x = $\frac{1}{2}(\underline{x} + \overline{x})$

and the radius of x, rad (x) = $\frac{1}{2}(\overline{x} - \underline{x})$

It can be proved (Neumaier, 1990) that

$\underline{x} \in \overline{x}$ if and only if

$\left|\overline{x} - \underline{x}\right| \le rad(x)$.

Like any other numerical type, the interval has a rich set of arithmetic operators associated with it, including binary operators for the basic operations of addition, subtraction, multiplication and division. Let x = $[\underline{x}, \overline{x}]$ and y = $[\underline{y}, \overline{y}]$ be two arbitrary operands of binary operator, and [a, b] be the result of applying the operator to its two operands, then the four basic arithmetic operators are denoted by;

Addition: $[\underline{x}, \overline{x}] + [\underline{y}, \overline{y}] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$

Subtraction: $[\underline{x}, \overline{x}] - [\underline{y}, \overline{y}] = [\underline{x} - \overline{y}, \overline{x} - \underline{y}]$

Multiplication: $[\underline{x}, \overline{x}] * [\underline{y}, \overline{y}]$ = [min , max]

where,

Min = minimum $(\underline{x} * \underline{y}, \underline{x} * \overline{y}, \overline{x} * \underline{y}, \overline{x} * \overline{y})$

Max = Maximum $(\underline{x} * \underline{y}, \underline{x} * \overline{y}, \overline{x} * \underline{y}, \overline{x} * \overline{y})$

Division $[\underline{x}, \overline{x}] * [\underline{y}, \overline{y}]$ $= [\underline{x}, \overline{x}] * [\frac{1}{\overline{y}}, \frac{1}{\underline{y}}]$

Provided that 0 ∉ $[\underline{x}, \overline{x}]$.

The uninary versions of addition and subtraction operators follow by noting that +

$[\underline{x}, \overline{x}] = [0, 0] + [\underline{x}, \overline{x}]$ and $-[\underline{x}, \overline{x}] = [0, 0] - [\underline{x}, \overline{x}]$

Where 0 denotes the additive identity value for the base type.

Improved rumps operation for interval circular arithmetic

In this section, we describe the interval matrix operation due to Rump [15] Assuming that we express an interval matrix A = $\hat{a} = ((\underline{a}+\overline{a})/2$ as the midpoint matrix and that r = $(\underline{a}-\overline{a})/2$ as the radius of the interval matrix. We define the entries of the mid point- radius interval matrix as a parametric notation

$$[a_{ij}] = \left\{\hat{a}, r\right\}, \left|\hat{a}\right| > r \text{ ,....................................... (3.1)}$$

We consider operations of two interval matrices using Rump's procedure (Rump, 2001 and 1999), assuming that A∈ $IR^{n\times n}$ and B∈ $IR^{n\times n}$ in the form.

$$A =< \hat{a}, r, > = \left\{a \in R : \hat{a} - r_1 \le a \le \hat{a} + r_2\right\}$$

$$for\ a \in R, 0 \le r_1 \in R \quad \text{and}$$

$$B =< \hat{b}, r_2 > = \left\{b \in R : \hat{b} - r_2 \le b \le \hat{b} + r_2\right\}$$

$$for\ b \in R,\ 0 \le r_2 \in R$$

Given that the four operations (+, -,.' /) for interval arithmetic are defined we have Rump's operations for two matrices as follows:

$$A+B = < a+b, r_1+r_2 >,$$
$$A-B = < a-b, r_1+r_2 >,$$
$$A\cdot B = < a\cdot b, |a|r_2 + |b|r_1 + r_1 r_2 >,$$
$$1/B = \left(\frac{b}{D}, \frac{r_2}{D}\right),\ where\ D = b^2 - r_2^2,$$
$$and\ 0 \notin B.$$
$$A/B = A\cdot\left(1/B\right),\ 0 \notin B.$$

We can decrease the excess width of the product A.B by using the ideas expressed in Ceberio and Kreinovich (2004) in the form as follows: We compute

mid (A.B) $=< \underline{a}.\overline{b} >$, its radius is given by r $= (\left|\overline{a}\right| + r)(\left|\overline{b}\right| + r) - \overline{a}\,\overline{b}$

We aim to accelerate the convergence behavior of Rump's operation by enlarging the disk to be inverted using ideas expressed in Carstensen and Petkovic (1994):

$$a^{-1} = \left\{\hat{a}, r\right\}^{-1} = \left\{\frac{1}{\hat{a}\left(1 - \frac{r^2}{\left|\hat{a}\right|^2}\right)}, \frac{r}{\left|\hat{a}\right|^2 - r^2}\right\}.. \quad (3.2)$$

$$a^{I_1} = \left\{\hat{a}, r\right\}^{I_1} = \left\{\frac{1}{\hat{a}}, \frac{r}{\left|\hat{a}\right|\cdot\left(\left|\hat{a}\right| - r\right)}\right\}.... \quad (3.3)$$

$$a^{I_2} = \left\{\hat{a}, r\right\}^{I_2} = \left\{\frac{1}{\hat{a}}, \frac{2r}{\left|\hat{a}\right|^2 - r^2}\right\}. \quad (3.4)$$

It is to be noted that $a^{-1} \subseteq a^{I_1} \subseteq a^{I_2}$

We also note that of the three inversions only $(a, r)^{-1}$ is exact operation. This means that $(\hat{a}, r)^{-1} = (\hat{a}^{-1} : a \in \hat{a})$ and that mid (a^{-1}) ≠ mid $(a)^{-1}$ in all cases. We will only adopt the disk inversion given in (3.2) for our purpose.

Following Petkovic and Vranic (2000) see also Carstensen and Petkovic (1994) it can be proved that

$$\left|mid(Inv(d))\right| \le \frac{|a|}{|a|^2 - r^2}, \text{...........(3.5)}$$

$$rad(Inv(d) \le \frac{2r}{|a|^2 - r^2} ....(3.6)$$

The disk inversions discussed above will be used to accelerate the basic iterative schemes to be discussed in section 4.

RESULTS

One main procedure for solving system (1.1) is to transform to the equivalent linear system (1.3) employing only interval arithmetic operations. We will denote A (x) to represent the Jacobian matrix $F(x)^{/}$ and b as representing the F(x) so that system (1.3) will take the equivalent form

Ad=-b (4.1)

Where d is the solution to linear interval equation (4.1). Then the splitting matrix A (x), will be decomposed in the form

A(x) = D(x) - L(x) - U(x),

Where D(x), L(x), U(x) are respectively, strictly diagonal matrix lower triangular matrix, and strictly upper triangular matrix. We will also assume that the matrix A(x) is regular diagonally dominant and positive definite.

Assuming that A(x) is ill conditioned or singular, one can overcome this in a satisfactory way whereby extended interval arithmetic can be used. In this circum-stance one has to result to box splitting and description of topological degree of that function. Detailed discussion of topological degree can be found in Kearfott (1998); Kearfott and Shi (2000); Kearfott and Hongthong (2005) and several others.

As an example, assuming that we set

$$x^{(k+1)} = x^{(k)} - \frac{(F(x))}{(F^{/}(x))} ....(4.2)$$

as representing Newtonian iteration where the operational weight signifies Newton correction given that 0 ∈ $F_i^{/}(x)$ and F(x) > 0, then Kahan arithmetic (see Kearfott and Shi (1996) gives

$$\frac{(F(x))}{(F^{/}(x))} = \left((-\infty, \frac{(F(x))}{(F^{/}(x))}\right) \cup \left(\frac{(F(x))}{(F^{/}(x))}, \infty\right) ....(4.3)$$

which can be used to decrease excess width of a box. The algorithms to be used in our discussion uses interval Gaussian elimination Algorithm IGA and interval Gauss-Siedel method. The Gaussian elimination algorithm from which IGA can be computed is given by

For k = 1 (1) n-1do

begin:

For i = k + 1 (1)n do

begin:

$$For \;\; j = k + 1(1)n \quad do$$

$$[a]ij^{(k+1)} = [a.]ij^{(k)} - \frac{[a]_{ik}^{(k)}}{[a]_{kk}^{k}} [a]_{kj}^{(k)}$$

$$[b]_i^{(k+1)} = [b]_i^{(k)} - \frac{[a]_{ik}^{(k)}}{[a]_{kk}^{k}} .[b]_k^{(k)}$$

$$end$$

$$for \quad l = 1(1)k \quad do$$

$$[a]_{ij}^{(k+1)} = [a]_{ij}^{(k)}$$

$$[b]_j^{k+1} = [b]_j^{k}$$

$$For \quad j = l + 1(1)n \quad do$$

$$[a]_{jl}^{(k+1)} = 0$$

$$end$$

$$end$$

$$[d]_n = \frac{[b]_n^n}{[a]_{nn}^n}$$

$$For \; i = n - 1(-1)1 \quad do$$

$$[d]i = [b]_i^n - \frac{\sum_{j=i+1}^{n} [a]_{ij}^n \cdot [d]_j}{[a]_{ii}^n}$$

It is assumed that the necessary pivoting had been performed to the matrix ahead so as to prevent division by an interval, which contains zero. We remark that in the algorithm given above we have not taken into account exchanges of rows or columns.

The feasibility of using IGA depends on interval maxtrix $A \varepsilon IR^{nxn}$ being regular. Thus for a general matrix A, problems are bound to occur, especially if the radii of the elements are too large. On the other side, undermining that IGA is not being effective in general applications, it has been found suitable for certain classes of matrices from which realistic bounds for the solution sets for M-matrices, H-matrices, diagonally dominant matrices, tridiagonally matrices have been reported to be adequate, Hargreaves (2002).

Alternative methods of interval Jacobi and interval Gauss-sieded types can be used for the enclosure of the solution set of the linear system (1.3). Their structural forms are given by interval Jacobi iteration

$$[d_i]^{(m+1)} = \left([b_i] - \sum_{j=1}^{n} [a_{ij}][d_j]^{(m)} \right) / [a_{ii}], i = 1,2,....n \quad (4.4)$$

and interval Gauss-Siedel iteration

$$[d_i]^{(m+1)} = \frac{1}{[a_{ii}]} \left([b_i] - \sum_{j=1}^{n} [a_{ij}][d_j]^{(m+1)} - \sum_{j=i+1}^{n} [a_{ij}][d_j]^{(m)} \right) (i = 1,2,....n) \quad (4.5)$$

We thus synchronize the interval Gaussian elimination method, the interval Jacobi iteration method and the interval Gauss-Siedel method, assuming it is known that IGA exists by the relation

$$[d]^{(m+1)} = \phi([d^{(m)}] \quad (4.6)$$

With

$$\phi([d]^{(m)}) = \text{IGA}([M], [N][d] + [b] \quad (4.7)$$

Where $[A] = [M] – [N]$

The Hierarchy with respect to generality can be obtained as follows: For [M] = [D], we easily obtain the Jacobi method. For [M] = [D] – {L] we have the Gauss-Siedel method. It is known that IGA exists for which the solution to the linear system can be obtained. We will only be interested in the usage of Gauss-Siedel method since it is faster than Jacobi iterative method.

The starting point for Newton method is the fulfillment that the iterates are approaching

x * successively such that

$$\left\| x^{(k+1)} - x^* \right\| < \left\| x^{(k)} - x^* \right\|, if\ x^* \neq x^* ....(4.8.)$$

$$where\ \left\| x^{(k)} - x^* \right\|\ is\ small\ enough.$$

However, this is without some problems x* is always not known apriori, an alternative criterion may then be provided. Thus instead of using the norm of the error, we can result to the measurement of the defect.

$$T(x) = \frac{1}{2} \left\| F(m(x))^2 \right\| = \frac{1}{2} F(m(x))^T F(m(x)), \quad (4.8)$$

The procedure for using (4.8) to control the iteration steps of Newton method has some useful advantages. This is due to the fact that F (m(x)) = 0 if and only if T(x) = 0, implies that x = x*. Note that we used m(x) to represent the midpoint vector x. It has been established that the choice of taking m(x) is optional which of course has been found to be optimal in some sense Alefeld and Herzberger (2000). It follows that the monotonicity requirement,

$T(x^{(k+1)}) < T(x^{(k)})$ holds

This inspires the following definition (Deuflhard, 1974), Let $\Delta(x)$denote the correction vector given in $x^{(k)}$ by the iterative method for solving systems of nonlinear equations. Then a level function $T(x/A)$ is said to be "appropriate" for the iterative method in question, if and only if

$\Delta (x_k)^T\ grad\ T(x^{(k)} (A) < 0$ for all $x^{(k)} \in D$ with $\Delta (x_k) \neq 0$.

We note that a level set is that set given by (4.7). In the cases of non singular Jacobian, that is, $F(x) \neq 0$, let the Newton correction vector $\Delta (x_v)$ be given by

$\Delta (x_k) = F^1 (x_k)^{-1} F(x_k),$

It is easy to see (Deuflhard, 1974) that

$\Delta (x_k)^T grad\ T (x_k \backslash A) = -2\ T (x_k \backslash A) < 0.$

Numerical illustration

We consider Uwamusi (2004a and 2004b)

$$f(x) = \begin{cases} 3x_1 - Cos(x_2 x_3) - 0.5 = 0 \\ x_1^2 - 81(x_2 + 0.1)^2 + Sin\ x_3 + 1.06 = 0 \\ e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0 \end{cases} \quad (5.1)$$
$$x^{(o)} = (0.1, 0.1, -0.1)^T$$

We assume that the given system (5.1) starting with initial vector x is contaminated with error of measurements or inaccurate data using ε-inflation (-ε,ε), where $\varepsilon = 10^{-2.}$ We have thus converted the system (5.1) to interval nonlinear system of equations which we can transform to the equivalent interval linear system (1.3) via Newton process

Table 1. Accelerated Rump's Newton Gauss-Siedel Method.

Iteration k	Mid (X_k)	Rad (X_k)
1	0.499871883	0.000032560
	0.019259157	0.0043193951
	-0.521732076	0.000266393
2	0.499893203	0.000055144
	0.001517517	0.004962856
	-0.523769576	0.000283299
3	0.5000000105	0.000055876
	0.00000062	0.00503736
	-0.523733712	0.0002834978

Table 2. Accelerated Rump's Newton Gaussian Algorithm (IGA) Method.

Iteration k	Mid (X_k)	Rad (X_k)
1	0.499870732	0.000032782
	0.019149969	0.004331234
	-0.525335038	0.000458863
2	0.500014059	0.000033752
	0.001524115	0.001073994
	-0.523320083	0.00046177
3	0.4999987251	0.000176312
	0.000000613	0.000477841
	-0.523696364	0.000462312

cess whose coefficients are now expressed as interval uncertainty. We solve the above problem to obtain the following results presented in the form of Tables 1, 2 and 3.

Conclusion

The provided numerical results have been obtained using Rump's operations for the midpoint-radius matrix and vector addition, subtraction, multiplication and division in conjunction with the procedure given by Ceberio and Kreinovich (2004). We used (Carstensen and Petkovic, 1994) circular arithmetic to enlarge the disks to be inverted in order to accelerate the basic interval algorithms. We halt the iteration when the value of $|F(m(x))| \leq 10^{-6}$ which implies that condition established in (3.8) is satisfied.

The presented method provided the worst bound case. It calculates guaranteed bounds on the true worst case performance range in every iteration. Thus the results are guaranteed to meet the specifications over the whole operating range. This is no surprising since the convergence of the midpoints and radii are coupled. We took the vector x for the given problem (5.1) to be in radian. We must stress here that one principal objective in this paper has been met. That is, to improve the Rump's interval matrix radius operation following Carstensen and Petkovic (1994), corroborated by Ceberio and Kreinovich (2004) procedures. It was also observed that the results obtained in Table 2 appeared to have significantly decreased in the values of radii as compared to those in Table 1 and Table 3. Also the obtained values in Table 1 gave better results than those from Table 3.This further showed that our presented method significantly improved that of Rump (1999). This is not to say that the worst case estimation methods based on standard real arithmetic do not have their own short comings. We observed in all the methods described above that the radii tend to grow instead of tending to zero. This may be due to the definition of midpoint –radius vector subtraction given in Carstensen and M.S. Petkovic (1994); Rump (1999) and Uwamusi (2004) and the references contained therein. The exact zeros for the problem (5.1) in point arithmetic are (0.4999999999, 0, -0.523696364). It thus appears that further investigation in future work may be necessary why the radii are not zeros.

Table 3. Rump's Method Using Newton Gauss-Siedel Iteration.

Iteration k	Mid (X_k)	Rad (X_k)
1	0.499734537	0.000040596
	0.019242743	0.004066086
	-0.521738767	0.000264917
2	0.499952391	0.000048532
	0.00220339	0.004670889
	-0.523767489	0.000248913
3	0.499999779	0.000048532
	0.000010264	0.004772143
	-0.523754595	0.000249186

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