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A collection of new preconditioners for solving linear systems

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In this paper, new preconditioners from class of (I+S)-type for solving linear systems are developed and preconditioned, accelerated overrelaxation (AOR) method is used for the systems. The proposed algorithms have a simple and graceful structure. Asymptotic convergence of the sequence generated by the methods to the unique solution of linear systems is proved, along with a result regarding the convergence rate of the preconditioned methods. Finally, computational comparisons of the standard methods against preconditioned methods based on examples are presented which illustrate the merits of simplicity, power and effectiveness of the proposed algorithms.

Key words: Preconditioning, accelerated overrelaxation (AOR), successive over relaxation (SOR), Z-, M-matrix.

INTRODUCTION

Science history indicates that substantial improvements and huge jumps in science and technology require interaction between mathematicians with different scientists. Meanwhile, solving linear equation system play the role of a catalyst for further connection of this interaction between mathematics and the sciences. Consider the following linear system:

$$Ax = b, \tag{1}$$

Where $A \in R^{n\times n}$, $b, x \in R^n$, we proposes to use the iterative method as:

$$x^{i+1} = M^{-1}Nx^i + M^{-1}b \qquad i = 0,1,\dots \tag{2}$$

For splitting A = M - N with the nonsingular matrix M. Lets consider the following preconditioned linear systems:

$$PAx = Pb \qquad p \in R^{n\times n}$$
$$or$$
$$AFY = b \qquad ; x = FY \tag{3}$$

Where p and F are respectively called left and right preconditioners. Therefore we have:

$$x^{i+1} = M_p^{-1}N_p x^i + M_p^{-1}pb \qquad i = 0,1,\dots$$

Where $PA = M_P - N_P$ and M_P is nonsingular. And

$$y^{(i+1)} = M_F^{-1}N_F y^{(i)} + M_F^{-1}b \qquad i = 0,1,..$$

Where $AF = M_F - N_F$ and M_F is nonsingular.

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These kind of preconditioned methods have been discussed and used by many researchers [Milaszewicz, 1987;

Gunawardena et al., 1991; Kotakemori et al., 1997, 2001; Kohno and Kotakemori, 1997; Evans et al., 2001; Sun, 2006; Wang, 2006; Wang and Song, 2009; Najafi and Edalatpanah (2011a, 2011b, 2012)].

For example, Milaszewicz (1987) presented the preconditioner (I + S), where the elements of the first column below the diagonal of A eliminate. Consider:

$$\hat{P} = I + \hat{S}$$

Where

$$\hat{S} = (s_{i,j}) = \begin{cases} -a_{i,j} & for \quad j = i+1, i = 1,2,\ldots,n-1 \\ 0 & for \quad otherwise \end{cases} \qquad (4)$$

Then $\hat{A} = (I + \hat{S})A$ can be written as follows:

$$\hat{A} = I - \hat{D} - L - \hat{E} - (U - \hat{S} + \hat{S}U)$$

Where $\hat{D}, \hat{E}$ are the diagonal and strictly lower triangular parts of $\hat{S}L$, respectively and also A = I – L - U, where I, is the identity matrix, L and U are strictly lower and strictly upper triangular matrices of A, respectively.

Gunawardena et al. (1991) considered the preconditioner ($I + S'$), which eliminates the elements of the first upper diagonal. Kohno and Kotakemori (1997) extended Gunawardena et al. (1991) work to a more general case as:

$$(1 + s_\alpha)$$

Where,

$$s_\alpha = (s_{ij})_{n\times n} = \begin{cases} -\alpha_i a_{ij} & j = i+1 \quad ,0 \le \alpha_i \le 1 \\ 0 & otherwise \end{cases}$$

Kotakemori et al. (2002) used:

$$\tilde{P} = I + S_{\max} \qquad (5)$$

Where

$$S_{\max} = (s_{i,j}) = \begin{cases} -a_{i,K_i} & for \quad i = 1,2,\ldots,n-1,\ j > i \\ 0 & for \quad otherwise \end{cases}$$

$$and \qquad k_i = \min \quad j \in \left\{ j \;\middle|\; \max_j |a_{ij}| \right\} \qquad for \quad i < n.$$

Then $\tilde{A} = (I + S_{\max})A$ can be written as:

$$\tilde{A} = I - \tilde{D} - L - \tilde{E} - (U - S_{\max} + \tilde{F} + S_{\max}U)$$

Where $\tilde{D}, \tilde{E}, \tilde{F}$ are the diagonal, strictly lower and strictly upper triangular parts of $S_{\max}L$, respectively. We also present some new preconditioners, by using the aforestated preconditioners.

THE AOR ITERATIVE METHOD FOR NEW PRECONDITIONER

Consider the preconditioned form of (3) as:

$$\bar{A}x = \bar{b} \qquad (6)$$

Where

$$\bar{A} = (1 + \bar{s}_\alpha + \bar{s}_k)A$$
$$\bar{b} = (1 + \bar{s})b$$
$$\bar{s} = \bar{s}_\alpha + \bar{s}_k$$
$$k \ge 1$$

and $\bar{s}_k = \begin{bmatrix} & 0\cdots & 0 \\ & 0.... & 0 \\ & \vdots....... & \vdots \\ \frac{-a_{n,1}}{k} & & \cdots 0 \end{bmatrix}$ and

$$\bar{s} = \begin{bmatrix} 0 & -\alpha_1 a_{12} & 0 & & 0 \\ 0 & 0 & -\alpha_1 a_{23} & & 0 \\ \vdots & & & & \\ 0 & 0 & & & -\alpha_{n-1}a_{n-1,n} \\ \frac{-a_{n,1}}{k} & 0\cdots & & & 0 \end{bmatrix}$$

Let A be split as:

A = I – L – U =

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} - \begin{bmatrix} 0 & & & & \\ -a_{21} & 0 & & & \\ -a_{31} & -a_{32} & 0 & & \\ \vdots & \vdots\ddots & & \ddots & \\ -a_{n1} & -a_{n2} & \cdots & -a_{n-1n} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ & 0 & -a_{23} & \cdots & -a_{2n} \\ & & \ddots & \ddots & \vdots \\ & & & 0 & \vdots \\ & & & & 0 \end{bmatrix}$$

Where I, -L, -U are respectively identity, strictly lower and upper triangular matrices.

Now we consider the splitting of (6) as:

$$\bar{A} = \bar{D} - \bar{L} - \bar{U}$$

Where,

$$\overline{D} = \begin{bmatrix} 1-\alpha_1 a_{12}a_{21} & & & \\ & 1-\alpha_2 a_{23}a_{32} & & \\ & & \ddots & \\ & & & 1-\dfrac{a_{1,n}a_{n,1}}{k} \end{bmatrix} \quad (7)$$

$$\overline{L} = \begin{bmatrix} 0 & & & & \\ \alpha_2 a_{23}a_{31} - a_{21} & 0 & & & \\ \vdots & \ddots & \ddots & & \\ (\frac{1}{k}-1)a_{n,1} & \frac{a_{n,2}a_{1,2}}{k} - a_{n,2} & \cdots & \frac{a_{n,1}a_{1,n-1}}{k} - a_{n,n-1} & 0 \end{bmatrix} \quad (8)$$

$$\overline{U} = \begin{bmatrix} 0 & (\alpha_1-1)a_{12} & \alpha_1 a_{12}a_{23} - a_{13}\cdots & \alpha_1 a_{12}a_{2n} - a_{1n} \\ 0 & 0 & (\alpha_1-1)a_{23}\ldots & \alpha_2 a_{23}a_{3n} - a_{2n} \\ \vdots & \vdots\ddots\ddots & 0\ddots & \vdots\ddots(\alpha_{n-1}-1)a_{n-1,n} \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \quad (9)$$

Generally,

$$\overline{A} = \begin{cases} a_{i,j} - \alpha_i a_{i,i+1}a_{i+1,j} & 1 \le i < n \\ a_{i,j} - \dfrac{a_{i,1}a_{1,j}}{k} & i = n \end{cases} \quad (10)$$

$$\overline{D} = \begin{cases} 1-\alpha_i a_{i,i+1}a_{i+1,i} & 1 \le i < n \\ 1-\dfrac{a_{1,i}a_{i,1}}{k} & i = n \end{cases} \quad (11)$$

In 1987, the accelerated over-relaxation iterative method (AOR) in Hadjidimos (1978) was defined as:

$$x^{(i+1)} = L_{r,w}x^{(i)} + (I-rL)^{-1}wb \qquad i = 0,1,\cdots \quad (12)$$

With iterative matrix:

$$L_{r,w} = (I-rL)^{-1}[(1-w)I + (w-r)L + wU] \quad (13)$$

Where (w, r) are real parameters with $w \neq 0$. Then, the iterative matrix with preconditioner of (6) is defined as:

$$\overline{L}_{r,w} = (\overline{D} - r\overline{L})^{-1}[(1-w)\overline{D} + (w-r)\overline{L} + w\overline{U}] \quad (14)$$

REQUIRED DEFINITIONS AND LEMMAS

Definition 1: [Berman and Plemmons (1994), Poole and Boullion (1974) and Najafi and Edalatpana (2011)]:

(a) The matrix A = $[a_{ij}]$ is nonnegative (positive) if $\forall\, i,j$ $a_{ij} \geq 0$($a_{ij} > 0$). In this case we write A ≥ 0(A>0). Similarly, for n-dimensional vectors *x*, by identifying them with n×1matrices, we can also define x ≥ 0 (x>0);
(b) A matrix *A* = a_{ij} is called a Z-matrix if for any $i \neq j, a_{ij} \leq 0$;
(c) A Z-matrix is a nonsingular M-matrix, if *A* is nonsingular, and if $A^{-1} \geq 0$.

Definition 2: A matrix A is irreducible if the directed graph associated to A is strongly connected (Varga, 2000).

Definition 3: Let *A* be a real matrix (Varga, 2000; Woznicki, 2001). The representation A = M – N is called a splitting of *A*, if *M* is a nonsingular matrix. The splitting is called:

(a) Convergent if $\rho\,(M^{-1}N)$ <1; (we denote the spectral radius of A by ρ(A));
(b) Regular if $M^{-1} \geq 0$, $N \geq 0$;
(c) Nonnegative if $M^{-1}N \geq 0$;
(d) M-splitting if *M* is a nonsingular M-matrix and $N \geq 0$;

Clearly, an M-splitting is regular and a regular splitting is non-negative.

Definition 4: A square matrix A = $(a_{ij})_{n\times n}$ is called (nonsingular) *M*-matrix (Berman and Plemmons, 1994), if

$A = \alpha I - B$; $B \geq 0$ And ($\alpha > \rho(B)$) $\alpha \geq \rho(B)$

Lemma 1: Let $A \in R^{n\times n}$ be nonnegative and irreducible n × n matrix (Varga, 2000; Theorem 2.7). Then,

(i) A has a positive real eigenvalue equal to its spectral radius $\rho(A)$;
(ii) For $\rho(A)$, there corresponds an eigenvector x >0;
(iii) $\rho(A)$ is a simple eigenvalue of A; and
(iv) $\rho(A)$ increases when any entry of A increases.

Lemma 2: (Li and Sun, 2000) Let A = M − N be an M-splitting of A. Then ρ ($M^{-1}N$) < 1 if and only if A is a nonsingular M-matrix.

Lemma 3: (Berman and Plemmons, 1994; Theorem 2.2]. Let A be a non-negative matrix. Then:

(1) If $\alpha x \le Ax$ for some non-negative vector x, $x \ne 0$, then $\alpha \le \rho(A)$;
(2) If $\alpha x \le \beta x$ for some positive vector x, then $\rho(A) \le \beta$. Moreover, if A is irreducible and if $0 \ne \alpha x \le Ax \le \beta x, \alpha x \ne Ax, Ax \ne \beta x$ for some non-negative vector x, then $\alpha < \rho(A) < \beta$ and x is a positive vector.

Lemma 4: (Berman and Plemmons, 1994, Theorem 6-2.3] Let A be a Z-matrix. Then A is a nonsingular M-matrix if and only if there is a positive vector x such that Ax>0.

Lemma 5: (Varga, 2000; Theorem 3.16). If $A \ge 0$ is a $n \times n$ matrix, then the following are equivalent:

(1) $\alpha > \rho(A)$;
(2) $\alpha I - A$ Is nonsingular, and $(\alpha I - A)^{-1} \ge 0.$

COMPARISON THEOREM

Let A be a Z-matrix, therefore, L ≥ 0 and U ≥ 0. Then we have:

$$(1-rL)^{-1} = 1 + rL + r^2L^2 + \cdots + r^{n-1}L^{n-1}$$
$$\longrightarrow L_{r,w} = (1 + rL + \cdots + r^{n-1}L^{n-1})[(1-w)I + (w-r)L + wU]$$
$$= (1-w)I + (w-r)L + wU + rL(1-w)I$$
$$+ rL\{(w-r)L + wU\} + (r^2L^2 + \cdots + r^{n-1}L^{n-1})[(1-w)I + (w-r)L + wU]$$
$$= (1-w)I + w(1-r)L + wU + \tau$$
$$where, \tau = rL[(w-r)L + wU] + (r^2L^2 + \cdots + r^{n-1}L^{n-1})[(1-w)I + (w-r)L + wU] \ge 0$$

And this means that $L_{r,w} \ge 0$. Also, it is easy to see that $L_{r,w}$ is irreducible when A is irreducible. For $\overline{L}_{r,w}$ the proof is as follows:

If $\rho(L_{r,w}) < 1$ then by Lemma 2, A is a nonsingular M-matrix and from Lemma 4:

$$\exists \;\; x > 0 \;\; S.T \;\; Ax > 0 \Rightarrow (I + \bar{s})Ax > 0.$$

It is easy to see that $\bar{A}$ is a Z-matrix and then by Lemma 4, we conclude that $\bar{A}$ is a nonsingular M-matrix; so

$1 - \alpha_i a_{i,i+1}a_{i+1,i} > 0, \;\; 1 - (1/k)a_{n1}a_{1n} > 0.$ So by Equations (7) and (11), we have $\bar{D} > 0$ and if:

$$\rho(L_{r,w}) \ge 1 \text{ and } \begin{cases} \alpha_i \le 1 \,\&\, a_{i,i+1}a_{i+1,i} < 1 \\ k \ge 1 \,\&\, 0 < a_{1,n}a_{n,1} < k \end{cases}$$

We have the aforestated conclusion. Also by (8), (9), $\bar{L} \ge 0, \bar{U} \ge 0$. So:

$$\overline{L_{r,w}} = (\bar{D} - r\bar{L})^{-1}\left[(1-w)\bar{D} + (w-r)\bar{L} + w\bar{U}\right]$$

$$= (1 - r\bar{D}^{-1}\bar{L})^{-1}\left[(1-w)I + (w-r)\bar{D}^{-1}\bar{L} + w\bar{D}^{-1}U\right]$$
$$= (1-w)I + w(1-r)\bar{D}^{-1}\bar{L} + w\bar{D}^{-1}U + \bar{\lambda}$$

$$\bar{\lambda} = r\bar{D}^{-1}\bar{L}\left[(w-r)\bar{D}^{-1}\bar{L} + w\bar{D}^{-1}U\right]$$
$$+\left[r^2(\bar{D}^{-1}\bar{L})^2 + \cdots\right]\{(1-w)I + (w-r)\bar{D}^{-1}\bar{L} + w\bar{D}^{-1}U\;\;\} \ge 0$$

Thus, we have proved the following corollary.

Corollary 1: Let A and $\bar{A}$ are respectively the coefficients matrix of (1) and (6), if $0 \le r \le w \le 1, w \ne 0, r \ne 1$ and A be a nonsingular Z-matrix and $\begin{cases} \alpha_i \le 1 \,\&\, a_{i,i+1}a_{i+1,i} < 1 \\ k \ge 1 \,\&\, 0 < a_{1,n}a_{n,1} < k \end{cases}$, then the iterative matrices $L_{r,w}$ and $\overline{L}_{r,w}$ of AOR methods applied to the linear systems (1), (6), respectively, are nonnegative. Now we show the improvement of convergence for this new preconditioner.

Theorem 1: Let $L_{r,w}$ and $\overline{L}_{r,w}$ are the iterative matrices of (13), (14) of AOR method. If A is a Z-matrix which is nonsingular and irreducible and conditions of corollary (1) are satisfied. Then we have:

1) If $\rho(L_{r,w}) < 1 \Rightarrow \rho(\overline{L}_{r,w}) \le \rho(L_{r.w})$
2) If $\rho(L_{r,w}) = 1 \Rightarrow \rho(\overline{L}_{r,w}) = \rho(L_{r.w})$
3) If $\rho(L_{r,w}) > 1 \Rightarrow \rho(\overline{L}_{r,w}) \ge \rho(L_{r.w})$

Proof: If A is irreducible then by proof of corollary(1) ,$L_{r,w}$ is nonnegative and irreducible. Therefore from Lemma 1 (Perron-Frobenius), there exists a vector x>0 such that $L_{r,w}x = \lambda x, \;\; \lambda = \rho(L_{r,w})$ from (13) we have:

$$(I - rL)^{-1}[(1-w)I + (w-r)L + wU]x = \lambda x \tag{15}$$

$$\rightarrow [(1-w)I + (w-r)L + wU]x = \lambda(I - rL)x \tag{16}$$

$$\longrightarrow [(1-w-\lambda)I+(w-r+\lambda r)L+wU]x=0 \tag{17}$$

$$\longrightarrow [w(L+U-I)+(1-\lambda)I+r(\lambda-1)L]x=0 \tag{18}$$

$$\longrightarrow w(L+U-I)x=(\lambda-1)(1-rL)x=0 \tag{19}$$

Also from (6), we have:

$$\overline{A}x=b$$
$$\longrightarrow (1+s_\alpha+s_k)(I-L-U)=$$
$$I-L-U+(s_\alpha+s_k)-(s_\alpha+s_k)L-(s_\alpha+s_k)U=\overline{D}-\overline{L}-\overline{U} \qquad (*)$$

And,

$$\overline{D}=(I-D_2-D_3) \tag{20}$$

$$\overline{L}=(L-s_K+L_2+L_3) \tag{21}$$

$$\overline{U}=(U-s_\alpha+U_3) \tag{22}$$

Where,

$$s=s_\alpha+s_k \tag{23}$$

$$sL=D_2+L_2 \tag{24}$$

$$sU=D_3+L_3+U_3 \tag{25}$$

To continue, we used lemma 3. Then, we have:

$$\overline{L}_{r,w}x-\lambda x=(\overline{D}-r\overline{L})^{-1}\left[(1-w)\overline{D}+(w-r)\overline{L}+w\overline{U}\right]-\lambda x$$
$$=(\overline{D}-r\overline{L})^{-1}\left[(1-w)\overline{D}+(w-r)\overline{L}+w\overline{U}-\lambda(\overline{D}-r\overline{L})\right]x$$
$$=(\overline{D}-r\overline{L})^{-1}[(1-w-\lambda)\overline{D}+(w-r+\lambda r)\overline{L}+w\overline{U}]x$$

$$\xrightarrow{(20,21,22)}$$
$$=(\overline{D}-r\overline{L})^{-1}\ [(1-w-\lambda)(I-D_2-D_3)+(w-r+\lambda r)(L-s_k+L_2+L_3)$$
$$+w(U-s_\alpha+U_3)]x$$
$$=(\overline{D}-r\overline{L})^{-1}\ [(1-w-\lambda)(I)+(w-r+\lambda r)(L)+w(U)]x+$$
$$(\overline{D}-r\overline{L})^{-1}[-(1-w-\lambda)(D_2+D_3)+(w-r+\lambda r)(L_2+L_3-s_k)+w(U_3-s_\alpha)]x$$

$$\xrightarrow{(17)}=(\overline{D}-r\overline{L})^{-1}[-(1-w-\lambda)(D_2+D_3)+(w-r+\lambda r)(L_2+L_3-s_k)+w(U_3-s_\alpha)]x$$
$$=(\overline{D}-r\overline{L})^{-1}[(\lambda-1)(D_2+D_3)+w(D_2+D_3)+r(\lambda-1)(L_2+L_3-s_k)+w(L_2+L_3-s_k)+w(U_3-s_\alpha)]x$$
$$=(\overline{D}-r\overline{L})^{-1}[(\lambda-1)(D_2+D_3)+r(\lambda-1)(L_2+L_3-s_k)+w(D_2+D_3+L_2+L_3-s_k+U_3-s_\alpha)]x$$

$$\xrightarrow{(23,24,25)}=(\overline{D}-r\overline{L})^{-1}[(\lambda-1)(D_2+D_3)+r(\lambda-1)(L_2+L_3-s_k)+w(sL+sU-s\ )]x$$
$$=(\overline{D}-r\overline{L})^{-1}[(\lambda-1)(D_2+D_3)+r(\lambda-1)(L_2+L_3-s_k)+ws(L+U-I)]x$$

$$\xrightarrow{(19)}=(\overline{D}-r\overline{L})^{-1}[(\lambda-1)(D_2+D_3)+r(\lambda-1)(L_2+L_3-s_k)+(\lambda-1)s(1-rL)]x$$
$$=(\overline{D}-r\overline{L})^{-1}[(\lambda-1)(D_2+D_3)+r(\lambda-1)(L_2+L_3-s_k)+(\lambda-1)s-r(\lambda-1)sL]x$$

$$=(\overline{D}-r\overline{L})^{-1}[(\lambda-1)(D_2+D_3)+r(\lambda-1)(L_2+L_3-s_k-sL)+(\lambda-1)s]x$$
$$=(\overline{D}-r\overline{L})^{-1}[(\lambda-1)(D_2+D_3)+r(\lambda-1)(L_2-sL)+$$
$$r(\lambda-1)(L_3)+r(\lambda-1)(-s_k)+(\lambda-1)s]x$$

$$\xrightarrow{(23,24)}$$
$$=(\overline{D}-r\overline{L})^{-1}[(\lambda-1)(D_2+D_3)+r(\lambda-1)(-D_2)+r(\lambda-1)(L_3)+(1-r)(\lambda-1)(s_k)+(\lambda-1)s_\alpha]x$$
$$=(\overline{D}-r\overline{L})^{-1}[(\lambda-1)(1-r)(D_2)+(\lambda-1)(D_3)+r(\lambda-1)(L_3)+(1-r)(\lambda-1)(s_k)+(\lambda-1)s_\alpha]x$$
$$=(\overline{D}-r\overline{L})^{-1}[(\lambda-1)(1-r)(D_2+s_k)+(\lambda-1)(D_3+s_\alpha)+r(\lambda-1)(L_3)]x \tag{26}$$

$$where, D_2+s_k\geq 0\ \&\ D_3+s_\alpha\geq 0\ \&\ L_3\geq 0$$

Also since $r\geq 0$, $\overline{L}\geq 0$ this is obviously that, $\overline{M}=(\overline{D}-r\overline{L})$ is a Z-matrix and since $r\overline{D}^{-1}\overline{L}\geq 0, \rho(r\overline{D}^{-1}\overline{L})=0<1$ by definition 4, $(I-r\overline{D}^{-1}\overline{L})$ is a nonsingular M-matrix and by Lemma 5, $(I-r\overline{D}^{-1}\overline{L})^{-1}\geq 0$

Therefore,

$$\overline{M}=(\overline{D}-r\overline{L})=\overline{D}(I-r\overline{D}^{-1}\overline{L})$$
$$\longrightarrow \overline{M}^{-1}=(1-r\overline{D}^{-1}\overline{L})^{-1}\overline{D}^{-1}\geq 0$$

$\longrightarrow \overline{M}$ is nonsingular M-matrix.

Now, let $p = \overline{L_{r,w}}x - \lambda x = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$ and $\overline{L_{r,w}} = (\overline{l_{i,j}})_{n\times n}$.

Hence from (26) we have:

$$\begin{aligned} &if \quad \lambda < 1 \quad then \quad p \le 0 \\ &if \quad \lambda \ge 1 \quad then \quad p \ge 0 \end{aligned}$$

Thus, by Lemma 3, we obtain the required results.

OTHER PRECONDITIONERS

The preconditioners which are presented here, are sort of extension of the previous preconditioner. Therefore, the proof of improvement in the convergence process is similar as that one, but we show that the iterative matrices are nonnegative.

The second preconditioned form

In this case, the preconditioned form of $\tilde{A}x = \tilde{b}$ is:

$$\tilde{A} = (1+\tilde{s})A$$
$$\tilde{b} = (1+\tilde{s})b$$ and

$$\tilde{s} = \begin{bmatrix} 0 & -\alpha_1 a_{12} & 0 & & 0 \\ 0 & 0 & -\alpha_1 a_{23} & & 0 \\ \vdots & & & & \\ 0 & 0 & & -\alpha_{n-1}a_{n-1,n} & \\ \frac{-a_{n,1}}{k_1} & \frac{-a_{n,2}}{k_2} & \cdots\cdots\cdots & \frac{-a_{n,n-1}}{k_{n-1}} & 0 \end{bmatrix} \qquad (27)$$

By direct computation we obtain that:

$$\tilde{A} = \tilde{a}_{i,j} = \begin{cases} a_{i,j} - \alpha_i a_{i,i+1} a_{i+1,j} & 1 \le i < n \\ a_{i,j} - \sum_{f=1}^{n-1} \frac{a_{i,f} a_{f,j}}{k_f} & i = n \end{cases}$$

$$\tilde{D} = \begin{bmatrix} 1-\alpha_1 a_{12}a_{21} & & & \\ & 1-\alpha_2 a_{23}a_{32} & & \\ & & \ddots & \\ & & & 1-\sum_{i=1}^{n-1}\frac{a_{n,i}a_{i,n}}{k_i} \end{bmatrix}$$

$$\tilde{L} = \begin{bmatrix} 0 & & & & \\ \alpha_2 a_{23}a_{31} - a_{21} & 0 & & & \\ \vdots & \ddots & \ddots & & \\ \{[(\frac{1}{k_1}-1)a_{n,1}] + (\sum_{i=2}^{n-1}\frac{a_{n,i}a_{i,1}}{k_i})\} & \{[((\frac{1}{k_2}-1)a_{n,2})] + (\sum_{\substack{i=1\\ i\ne 2}}^{n-1}\frac{a_{n,i}a_{i,2}}{k_i})\} & \cdots & \{[((\frac{1}{k_{n-1}}-1)a_{n,n-1})](\sum_{i=1}^{n-2}\frac{a_{n,i}a_{i,n-1}}{k_i})\} & 0 \end{bmatrix}$$

$$\tilde{U} = \overline{U}$$

Obviously, $(\forall i, k_i \ge 1)$; $\tilde{U} \ge 0, \tilde{L} \ge 0$. Also if $\rho(L_{r,w}) < 1$ then, from proof of corollary 1, $\tilde{D} > 0$. Otherwise, under the following conditions elements of the diagonal are positive:

$$\begin{cases} \alpha_i \in [0,1] \,\& & 0 \le a_{i,i+1}a_{i+1,i} < 1 \\ \forall i : k_i \ge 1 \,\& & 0 \le \sum_{i=1}^{n-1}\frac{a_{n,i}a_{i,n}}{k_i} < 1 \end{cases}$$

The third preconditioned form

Consider the preconditioned system $\tilde{\tilde{A}}x = \tilde{\tilde{b}}$ as:

$$\tilde{\tilde{A}} = (1+\tilde{\tilde{s}})A$$
$$\tilde{\tilde{b}} = (1+\tilde{\tilde{s}})b$$

$$\tilde{\tilde{s}} = \begin{bmatrix} 0 & -\alpha_1 a_{12} & 0 & & -ta_{1,n} \\ 0 & 0 & -\alpha_1 a_{23} & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & & & -\alpha_{n-1}a_{n-1,n} \\ \frac{-a_{n,1}}{k_1} & \frac{-a_{n,2}}{k_2} & \cdots\cdots & \frac{-a_{n,n-1}}{k_{n-1}} & 0 \end{bmatrix} \quad (28)$$

By direct computation we obtain that:

$$\tilde{\tilde{A}} = \tilde{\tilde{a}}_{i,j} = \begin{cases} a_{i,j} - \alpha_i a_{i,i+1} a_{i+1,j} - ta_{1,n} a_{n,j} & i=1 \\ a_{i,j} - \alpha_i a_{i,i+1} a_{i+1,j} & 1<i<n \\ a_{i,j} - \sum_{f=1}^{n-1} \frac{a_{i,f} a_{f,j}}{k_f} & i=n \end{cases}$$

$$\tilde{\tilde{D}} = \begin{bmatrix} 1-\alpha_1 a_{12}a_{21} - ta_{1,n}a_{n,1} & & & \\ & 1-\alpha_2 a_{23}a_{32} & & \\ & & \ddots & \\ & & & 1-\sum_{i=1}^{n-1}\frac{a_{n,i}a_{i,n}}{k_i} \end{bmatrix} , \tilde{\tilde{L}} = \tilde{L}$$

$$\tilde{\tilde{U}} = \begin{bmatrix} 0 & (\alpha_1 -1)a_{12} + ta_{1,n}a_{n,2} & \alpha_1 a_{12}a_{23} + ta_{1,n}a_{n,1} - a_{13} & \cdots & (t-1)a_{1,n} + \alpha_1 a_{12}a_{2,n} \\ 0 & 0 & (\alpha_2 -1)a_{23}\cdots & & \alpha_2 a_{23}a_{3,n} - a_{2,n} \\ & & & \ddots & \vdots \\ \vdots & \vdots\ddots\cdots & 0 & & (\alpha_{n-1}-1)a_{n-1,n} \\ 0 & & & & 0 \end{bmatrix}$$

Obviously, $(\alpha_i, t \in [0,1] \,\&\, k_i \geq 1); \tilde{\tilde{U}} \geq 0, \tilde{\tilde{L}} \geq 0$. Also, if $\rho(L_{r,w}) < 1$ then, from proof of corollary 1, $\tilde{\tilde{D}} > 0$. Otherwise, under the following conditions, elements of diagonals are positive:

$$\begin{cases} \alpha_i, t \in [0,1], k_i \geq 1 \\ 0 \leq a_{12}a_{21} + ta_{1,n}a_{n,1} < 1 \\ 0 \leq a_{i,i+1}a_{i+1,i} < 1 \\ \sum_{i=1}^{n-1} \frac{a_{n,i}a_{i,n}}{k_i} < 1 \end{cases}$$

And with this process we can make up preconditioners by the following models; the 4th and 5th preconditioned forms (29), (30) respectively:

$$\bar{\bar{A}}X = \bar{\bar{b}}; \qquad \bar{\bar{s}} = \begin{cases} -\alpha_i a_{ij} & for (j=i+1), (j=n, i\neq n), (j\neq n, i=n) \\ 0 & otherwise \end{cases} \quad (29)$$

$$\tilde{\bar{A}}X = \tilde{\bar{b}}; \qquad \bar{\bar{s}} = \begin{cases} -\alpha_i a_{ij} & for (j=i+1), (j=n, i\neq n), (j\neq n, i=n), (j\neq 1, i=1) \\ 0 & otherwise \end{cases} \quad (30)$$

The sixth preconditioned form

Similar to Kotakemori et al. (2002), we consider the following preconditioner;

$$p = (I + S_{\min}) \quad (31)$$

Where

$$S_{\min} = \begin{cases} 0 & if \quad a_{i,j} \in a_{i,Q_i} \\ -a_{i,j} & otherwise \end{cases} \qquad i = 1 : n-1,\ j > i \quad (32)$$

Also, similar to k_i in $S_{\max}$, Q_i is given by the following:

$$Q_i = j \in \left\{ j \;\middle|\; \min_j |a_{ij}| \right\} \qquad for \quad i < n-1$$

Then, the AOR preconditioned matrix is as follows:

$$A_{\min} = (I + S_{\min})A,$$
$$A_{\min} = (I - L - U) + S_{\min} - S_{\min} L - S_{\min} U$$
$$\longrightarrow A_{\min} = \underbrace{(I - D)}_{D_{\min}} - \underbrace{(L + E)}_{L_{\min}} - \underbrace{(U - S_{\min} + F + S_{\min} U)}_{U_{\min}}$$

$$\Rightarrow \begin{cases} M_{\min} = D_{\min} - rL_{\min} \\ N_{\min} = (1-w)D_{\min} + (w-r)L_{\min} + wU_{\min} \end{cases} \quad (33)$$

Where D, E and F are the diagonal, strictly lower and strictly upper triangular parts of $S_{\min} L = D + E + F \geq 0$, respectively. It can be seen that:

$$A_{\min} = \begin{cases} a_{i,j} - \sum_{k \notin Q_i} a_{i,k} a_{k,j} & 1 \leq i < n \\ a_{i,j} & i = n \end{cases} \quad (34)$$

E,**F** are nonnegative and $D = I - \sum_{k \notin Q_i} a_{i,k} a_{k,i}$. So if

$$1 \neq \sum_{k \notin Q_i} a_{i,k} a_{k,i} \quad ; i = 1,2,\ldots,n-1,$$

Then $M_{\min}^{-1}$ exists.

Theorem 2: let $L_{r,w}$ be the iterative matrix of (13) and $\widehat{L}_{r,w}$ be an iterative preconditioned matrix of AOR method by any of our preconditioned models and A is an irreducible and nonsingular Z-matrix. If by the aforestated conditions $\widehat{L}_{r,w} \geq 0$. Then, we have:

1) If $\rho(L_{r,w}) < 1 \Rightarrow \rho(\hat{L}_{r,w}) \leq \rho(L_{r.w})$

2) If $\rho(L_{r,w}) = 1 \Rightarrow \rho(\hat{L}_{r,w}) = \rho(L_{r.w})$

3) If $\rho(L_{r,w}) > 1 \Rightarrow \rho(\hat{L}_{r,w}) \geq \rho(L_{r.w})$

Proof: Let $\widehat{A}$ be the preconditioned matrix of A with preconditioner of $(I + \hat{S})$. Without loss of generality, let $\hat{S} = (S_L + S_U)$. Where S_L, S_U are strictly lower and upper triangular matrices obtained from $\widehat{A}$, respectively. Then we have:

$$\hat{A} = (1 + S_L + S_U)A = I - L - U + S_L - S_L L - S_L U + S_U - S_U L - S_U U$$

let $\begin{cases} S_L U = D_2 + L_2 + U_2 \\ S_U L = D_3 + L_3 + U_3 \end{cases}$ and $\hat{A} = \hat{D} - \hat{L} - \hat{U}$ where,

$$\begin{cases} \hat{D} = (I - D_2 - D_3) \\ \hat{L} = L - S_L + S_L L + L_2 + L_3 \\ \hat{U} = U + U_2 - S_U + U_3 - S_U U \end{cases}$$

Now, with proof of Theorem 1, we have:

$$\hat{L}_{r,w}x - \lambda x = (\hat{D} - r\hat{L})^{-1}[(1 - w - \lambda)\hat{D} + (w - r + \lambda r)\hat{L} + w\hat{U}]x$$
$$= (\hat{D} - r\hat{L})^{-1}[(\lambda - 1)(D_2 + D_3) + r(\lambda - 1)(L_2 + L_3 - S_L + S_L L) + (\lambda - 1)\hat{s}(1 - r\hat{L})]x$$
$$= (\hat{D} - r\hat{L})^{-1}[(\lambda - 1)(D_2 + D_3) + r(\lambda - 1)(L_2 + L_3)$$
$$+ r(\lambda - 1)(-S_L + S_L L) + (\lambda - 1)(S_L + S_U) - r(\lambda - 1)(S_L L + S_U L)]x$$

Table 1. The results of example 1.

w	r	ρ	$\hat{\rho}$	$\tilde{\rho}$	$\bar{\rho}$
1	0	0.7352	0.6774	0.6709	0.5890
0.9	0.4	0.7188	0.6549	0.6519	0.5696
0.9	0.5	0.7058	0.6369	0.6351	0.5501
0.9	0.6	0.6912	0.6163	0.6160	0.5279
0.9	0.8	0.6553	0.5635	0.5679	0.4716
1	1	0.5604	0.4210	0.4380	0.3119

$$=(\hat{D}-r\hat{L})^{-1}[(\lambda-1)(D_2+D_3)+r(\lambda-1)(L_2+L_3)+(1-r)(\lambda-1)(S_L)+(\lambda-1)(S_U)+r(1-\lambda)(S_U L)]x$$

On the other hand by (17), we have:

$$L=\frac{-(1-w-\lambda)I-wU}{(w-r+\lambda r)}$$

Therefore,

$$\hat{L}_{r,w}x-\lambda x=$$

$$(\hat{D}-r\hat{L})^{-1}[(\lambda-1)(D_2+D_3)+r(\lambda-1)(L_2+L_3)+(1-r)(\lambda-1)(S_L)+(\lambda-1)w\frac{\{(1-r)S_U+S_U U\}}{w-r+\lambda r}]x$$

Thus by Lemma 3, we obtain the required results.

Remark 2: In (13) by choice special parameters can be obtained, similar results about popular method from the aforestated theorems. Evidently:

1) Jacobi method for w = 1, r = 0.
2) JOR (Jacobi Over relaxation) method for r = 0.
3) Gauss-seidel method for r = w = 1.
4) SOR method for r = w.

NUMERICAL EXAMPLES

Here, we give two examples to illustrate the results obtained earlier.

Example 1: The coefficient matrix *A* of (1) is given by:

$$A=\begin{pmatrix} 1 & -0.2 & -0.023 & -0.18 & -0.27 & -0.031 & -0.1 \\ -0.1 & 1 & -0.31 & -0.18 & -0.07 & -0.1 & -0.2 \\ -0.01 & -0.1 & 1 & -0.1 & -0.2 & -0.17 & -0.0098 \\ -0.021 & -0.2 & -0.03 & 1 & -0.3 & -0.01 & -0.1 \\ -0.1 & -0.014 & -0.09 & -0.03 & 1 & -0.1 & -0.1 \\ -0.2 & -0.023 & -0.1 & -0.27 & -0.03 & 1 & -0.1 \\ -0.18 & -0.0081 & -0.1 & -0.019 & -0.1 & -0.2 & 1 \end{pmatrix}$$

If we apply all the last methods for A and compute the spectral radius in each case we have the following results. In Table 1, we reported the spectral radius of the corresponding iterative matrix with different parameters w, r. We denote spectral radius of the AOR method by ρ. Also $\hat{\rho}$, $\tilde{\rho}$, $\bar{\rho}$ are spectral radius of iteration matrix with preconditioners (4), (5) and (31), respectively.

Example 2: The coefficient matrix *A* of (1) is given by:

$$A=(a_{i,j})_{n\times n}=\begin{cases}1 & if \quad i=j \\ \dfrac{-1}{i+2j} & if \quad i\neq j\end{cases}$$

In Table 2, we reported the spectral radius of the corresponding iterative matrix with different parameters w, r. We denote spectral radius of the AOR method by $\rho(L_{r,w})$ and $\rho(L_{R,W})$; $\rho(\bar{L}_{r,w})$, $\rho(\bar{\bar{L}}_{r,w})$, $\rho(\tilde{\tilde{L}}_{r,w})$ are spectral radius of the Ganawardena's preconditioner, first, second, fourth and fifth preconditioners as stated earlier. Also, we take $k_i,\alpha_i,t=1$ in proposed preconditioners. From Table 1, we can see that all the

Table 2. The spectral radius of the AOR method with different preconditioners.

N	w	r	$\rho(L_{r,w})$	$\rho(L_{R,W})$	$\rho(\bar{L}_{r,w})$	$\rho(\tilde{L}_{r,w})$	$\rho(\bar{\bar{L}}_{r,w})$	$\rho(\tilde{\tilde{L}}_{r,w})$
8	1	0	0.6470	0.6028	0.5999	0.5875	0.5845	0.5170
	0.9	0.7	0.5794	0.5184	0.5137	0.5035	0.5001	0.4410
	0.9	0.8	0.5558	0.4886	0.4832	0.4735	0.4699	0.4119
	1	1	0.4341	0.3333	0.3241	0.3151	0.3100	0.2492
16	1	0	0.8271	0.8095	0.8092	0.8065	0.8080	0.7745
	0.9	0.7	0.7771	0.7474	0.7467	0.7439	0.7431	0.7071
	0.9	0.8	0.7614	0.7276	0.7267	0.7238	0.7229	0.6853
	1	1	0.6886	0.6360	0.6346	0.6314	0.6298	0.5833
24	1	0	0.9210	0.9139	0.9138	0.9138	0.9130	0.8994
	0.9	0.7	0.8938	0.8806	0.8804	0.8795	0.8793	0.8617
	0.9	0.8	0.8855	0.8702	0.8700	0.8691	0.8688	0.8499
	1	1	0.8486	0.8244	0.8241	0.8230	0.8225	0.7978

numerical results have illustrated our theoretical analysis. For example the spectral radius of the classical Gauss-seidel and Ganawardena's preconditioner with N = 8, are 0.4341, 0.3333, while the spectral radius of our fifth preconditioner for Gauss-seidel is 0.2492.

CONCLUSION

In this paper, we have proposed some new preconditioners from the class of (I + S)-type based on the AOR method. Also, we let the coefficient matrix of linear system be Z-matrix or M-matrix that often occur in a wide variety of area including numerical differential equation, growth models in economics and physical and biological sciences (Berman and Plemmons, 1994).

Finally, from theorems and numerical examples, it may be concluded that the convergence rate of our proposed methods are superior to the basic AOR Method and better than some preconditioner of (I + S)-type.

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