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*Full Length Research Paper*

# Using state feedback control to stabilize unstable equilibrium points of the unified fractional-order chaotic system

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**In this paper, a state feedback controller is designed to stabilize a class of fractional order unified systems at their unstable equilibrium points. The fractional-order chaotic Chen system is chosen to get stabilized in two cases of commensurate and incommensurate orders. An observer based technique is used to identify an unknown parameter of the unified system. The system with identified parameter is stabilized at their unstable equilibrium points using two different methods of; a classical feedback scheme and so called the enhanced one. According to the Routh-Hurwitz and relevant stability theorem for fractional system, the stability criteria are discussed whilst they are theoretically proven. Simulation results are demonstrated for the Chen system to illustrate the effectiveness of the proposed control scheme. Simulation result verifies using the proposed control technique increases the stability region of the unified fractional system regardless for commensurate or incommensurate orders. Considering the simplicity in the structure of the controller, the convergence speed is found satisfactory.**

**Key words:** Unified chaotic system, state feedback, uncertainty principle, chaos stabilization, equilibrium points.

## INTRODUCTION

Chaos theory, as a new branch of physics and mathematics, has provided a new way of viewing the world. It gives an important tool to truly develop the traditional approach. Chaotic behaviors have been observed in different areas of science and engineering such as mechanics, electronics, physics, medicine, ecology, biology, and economy. To avoid problems arising from unusual behaviors of a chaotic system, chaos control has gained increasing attention. In recent years, a new direction of chaos research has emerged, in which fractional order calculus is applied to dynamic systems (Wang and Yu, 2008).

Fractional calculus, in essence is an extension of the classical calculus, with almost 300-year history. In spite of the long history, applications of the fractional calculus to physics and engineering have just attracted recent focus of interest (Podlounby, 1999). It has been found that the behaviour of many physical systems can properly be described using the fractional order system theory. For example, heat conduction (Jenson and Jeffreys, 1997), quantum evolution of complex systems (Kusnezov et al., 1999), and diffusion waves (El-Sayed, 1996) are known of such systems which are concerned by the fractional order equations. In fact, real world process is most likely of the fractional order system (Torvik and Bagley, 1984). Recently, there is a new trend to investigate the control and dynamics of fractional order dynamical systems. It is shown that nonlinear chaotic systems can behave chaotic when their models become fractional (Ahmad and Sprott, 2003). In (Ahmad and Harba, 2003), controllers have been designed using ‘‘backstepping’’ method of nonlinear control design. Investigation and control the chaotic behavior of fractional-order Coullet system, including the necessary condition for appearance of chaos is studied (Shahiri et

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al., 2010). In Li et al. (2003), chaos synchronization of fractional order chaotic systems is also investigated.

A unified chaotic system is a category of systems in which different chaotic attractors will be expressed by a different parameter (Wang and Song, 2008). Several researchers have focused on control and synchronization of the unified chaotic system. In Park et al. (2009) the problem of adaptive $H_\infty$ synchronization for unified chaotic systems with uncertain parameter and external disturbance is studied. Lü et al. (2004) used linear state feedback and adaptive control to synchronize an identical unified chaotic system with only one input controller. In (Park, 2005, 2007; Lee et al., 2010) based on the Lyapunov method, different controllers are designed to achieve a synchronization of unified systems. In some other papers, for example, (Junwei and Yanbin, 2006; Xian et al., 2008; Weihua and Changpin, 2008; Wang and He, 2008), fractional dynamic of unified systems has been studied. Chaotic behavior of fractional-order unified system is discussed in (Junwei and Yanbin, 2006; Xian et al., 2008; Weihua and Changpin, 2008). Whilst a synchronization of fractional-order unified system, using the Laplace transform and the final-value theorem, has also been reported (Junwei and Yanbin, 2006; Xian et al., 2008). In (Wang and He, 2008), a projective synchronization of fractional order unified system based on linear separation is studied.

In this paper, a systematic state feedback controller is used to asymptotically stabilize commensurate and incommensurate fractional-order unified chaotic systems. Primarily, based on observer identification technique, an uncertain parameter of the unified system is shown identifiable. The work will be followed when two different methods of; the classical feedback control and the enhanced feedback control methods, are used to stabilize unstable equilibrium points of the system. Stabilization methods are fulfilled in two steps of; locating the eigenvalues on the negative real axis and then locating them in the imaginary axis, of course in pair(s) of pole(s). Finally, Numerical simulations show the performance of the procedure.

## PRELIMINARIES

Among several definitions of fractional derivatives, the following Caputo-type definition is more of interest (Matignon, 1996):

$$
{}_0D_t^q f(t) = \begin{cases} \dfrac{1}{\Gamma(m-q)}\displaystyle\int_0^t \dfrac{f^m(\tau)}{(t-\tau)^{q+1-m}}\,d\tau, & m-1<q\le m \\ \dfrac{d^m}{dt^m} f(t) & q=m \end{cases} \tag{1}
$$

Where $D_t^q = \dfrac{d^q}{dt^q}$ and $m$ is the first integer number larger than $q$.

**Theorem 1** (Matignon, 1996): The following fractional-order system:

$$
\frac{d^{q_i}x_i}{dt^{q_i}} = f_i(x_1, x_2, \dots x_n), \quad i = 1, 2, \dots, n \tag{2}
$$

With $0 < q_i < 1, x \in R^n$ is asymptotically stable if and only if the following equation is met:

$$
\arg(\lambda) \ge \pi Q/2, \; Q = \max(q_i) \tag{3}
$$

Where $\lambda$ 's are the eigenvalues of the following Jacobian matrix *J*, at the equilibrium point.

$$
J = \begin{pmatrix} \partial f_1/\partial x_1 \cdots\cdots\cdots \partial f_1/\partial x_n \\ \vdots \quad\quad \ddots \quad\quad \vdots \\ \vdots \quad\quad\quad \ddots \quad \vdots \\ \partial f_n/\partial x_1 \cdots\cdots\cdots \partial f_n/\partial x_n \end{pmatrix}
$$

Figure 1 shows the corresponding the stability region of the fractional order systems (Matignon, 1996).

**Definition 1:** The same order of derivative in the state equation (2), i.e. $q_1 = q_2 \cdots = q_i$, constructs a commensurate order system, and vice versa for $(q_1 \ne q_2 \cdots \ne q_i)$ which is called an incommensurate order one.

**Theorem 2** (Tavazoei and Haeri, 2008): Consider an *n*-dimensional nonlinear incommensurate fractional order system in Eq. (2) in which all $q_i$ 's are rational numbers belonging to [0, 1]. Assume $M$ be the lowest common multiple of the denominators $u_i$ 's of $q_i$ 's, where $q_i = v_i/u_i$, which means $v_i, u_i \in Z^+$ have no common factor, for $i = 1, 2, \dots, n$. A necessary condition to create a chaotic attractor is mathematically equivalent to:

$$
(\pi/2M) - \min_i\{|\arg(\lambda_i)|\} \ge 0 \tag{4}
$$

Where; $\lambda_i$ 's are roots of the characteristic polynomial of:

$$
\det\left(\operatorname{diag}\left(\left[\lambda^{Mq_1} \quad \lambda^{Mq_2} \dots \lambda^{Mq_n}\right]\right) - \partial f/\partial x\big|_{x=x^*}\right) = 0
$$
$$
\forall x^* \in \Omega
$$

In which $\Omega$ is the set of equilibrium points of the system, surrounded by scrolls. The term $(\pi/2M) - \min_i\{|\arg(\lambda_i)|\}$ is called the Instability Measure for equilibrium points in Fractional Order Systems (IMFOS).

**Definition 2:** (Basu et al., 2003). The discriminant of a polynomial $f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots a_n$ is defined as:

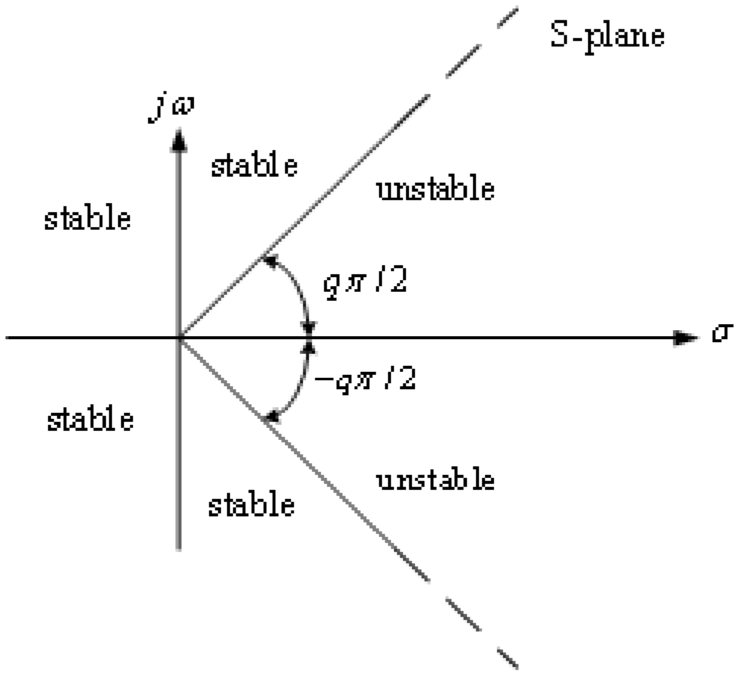


**Figure 1.** The stability region of fractional-order system with order $0<q<1$.

$$D(f)=\prod_{\substack{i,j\\ i<j}}^{n}(r_i-r_j)^2$$

Where $r_i$ and $r_j$ are the roots of the polynomial $f(x)$.

If $D(f)>0$ then $f(x)=0$ has even numbers of pair of complex roots. For $D(f)<0$ then equation $f(x)=0$ has odd numbers of pair of complex roots. Meanwhile for $n=3$, the discriminant of $f(x)$ is described as:

$$D(f)=18a_1a_2a_3+(a_1a_2)^2-4a_3a_1^3-4a_2^3-27a_3^2 \tag{5}$$

**Proposition 1** (Xiang-Yuan et al., 2009): For $n=3$, if the discriminant $D(P)$ of $P(\lambda)=\lambda^3+a_1\lambda^2+a_2\lambda+a_1$ is positive, then the Routh-Hurwitz conditions (Dorf and Bishop, 1957) i.e. $a_1>0, a_2>0, a_3>0, a_1a_2>a_3$, are the necessary and sufficient stability conditions for Eq. (3). In this case, eigenvalues are located in the negative $x-\text{axis}$.

**Proposition 2** (Xiang-Yuan et al., 2009): For $n=3$, if the discriminant $D(P)$ of $P(\lambda)=\lambda^3+a_1\lambda^2+a_2\lambda+a_1$ is negative, when $a_1>0, a_2>0, a_1a_2=a_3,\ \forall\theta\in[0,1)$, then all eigenvalues of $P(\lambda)=0$ satisfy Eq. (3). In this case, all complex eigenvalues are located in the imaginary axis.

## The system description

Lü in (Lü et. al., 2002) considered a class of unified chaotic form of the following:

$$\begin{cases}\dfrac{dx}{dt}=(25\alpha+10)(y-x)\\ \dfrac{dy}{dt}=(28-35\alpha)x-xz+(29\alpha-1)y\\ \dfrac{dz}{dt}=xy-\dfrac{8+\alpha}{3}z\end{cases} \tag{6}$$

Where $x_1,x_2,x_3$ are state variables and $\alpha\in[0,1]$ is a key parameter of the system. The system (6) is found chaotic for any $\alpha\in[0,1]$. When $\alpha=0, \alpha=0.8$ and $\alpha=1$, it is called the Lorenz, the Lü and the Chen chaotic attractors respectively. In order to construct a fractional order of the unified system, standard derivatives of equation (6) will be replaced by the following fractional differentiation:

$$\begin{cases}\dfrac{d^{q_1}x}{dt^{q_1}}=(25\alpha+10)(y-x)\\ \dfrac{d^{q_2}y}{dt^{q_2}}=(28-35\alpha)x-xz+(29\alpha-1)y\\ \dfrac{d^{q_3}z}{dt^{q_3}}=xy-\dfrac{8+\alpha}{3}z\end{cases} \tag{7}$$

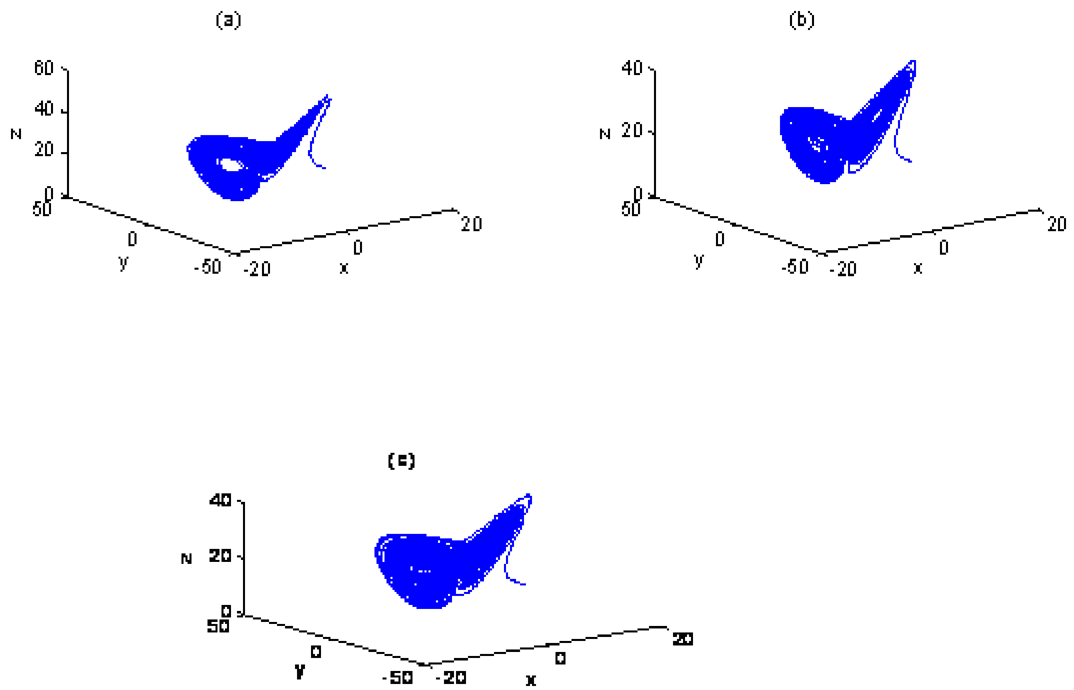


**Figure 2.** Chaotic attractors of the fractional order unified system (3) with $(q_1, q_2, q_3) = (0.85, 0.95, 0.9)$ : (a) $\alpha = 0.3$ (the Lorenz system), (b) $\alpha = 0.8$ ( the Lu system), (c) $\alpha = 1$ ( the Chen system).

Where $d^{q_i} / dt^{q_i} = D_t^{q_i}$, $(i = 1, 2, 3)$. The order $q = (q_1, q_2, q_3)$, is subjected to $0 < q_1, q_2, q_3 \leq 1$, specifically when $\alpha \in [0,1]$. However system (7) has three equilibrium points of:

$$O = (0, 0, 0)$$
$$C^{\pm} = (\pm\sqrt{(8+\alpha)(9-2\alpha)}\ , \pm\sqrt{(8+\alpha)(9-2\alpha)}\ , 27-6\alpha) \tag{8}$$

Where $O$ (Tavazoei and Haeri, 2008) includes a saddle point of index 1 and $C^{\pm}$ are of a saddle point of index 2. Therefore, all three equilibrium points of unified system are unstable. The chaotic behaviour of fractional order unified (the Chen, Lü and Lorenz) systems for $0.91 < q_1 = q_2 = q_3 \leq 1$ are plotted in (Xian et al., 2008). From theorem 2 will be shown that for a set of parameter $\alpha \in [0.3, 1]$ of $(q_1, q_2, q_3) = (0.85, 0.95, 0.9)$, the fractional-order unified system displays chaotic attractors. For $\alpha = 0.25$, the instability measurement will be found as IMFOS= $-9.2 \times 10^{-4} < 0$. Therefore, for order of $(q_1, q_2, q_3) = (0.85, 0.95, 0.9)$, the system in (3) will not yet comply with the necessary requirement. However, for:

$$\left.\begin{array}{l} \alpha\text{=0.3:IMFOS=}5.22\times10^{-5} > 0 \\ \alpha\text{=0.8:IMFOS=}6.72\times10^{-3} > 0 \\ \alpha\text{=1.0:IMFOS=}7.78\times10^{-3} > 0 \end{array}\right\} \Rightarrow$$

System (3) meets the necessary condition to show chaotic attractor when $(q_1, q_2, q_3) = (0.85, 0.95, 0.9)$. Numerical results which are depicted in Figure 2, illustrate the existence of the chaotic attractor for the given fractional orders.

## Identification of the unknown parameter α

In this study, an observer is designed to identify the key parameter in system (6) when the parameter of the system is assumed unknown. From the prior knowledge, parameter $b$ is assumed constant. This means:

$$\dot{b} = 0 \tag{9}$$

Where:

$$b = \frac{\alpha + 8}{3} \tag{10}$$

However, to estimate $b$, the following observer is proposed:

$$\dot{e} = kz\left[\hat{b}z - (xy - D^{q_3} z)\right] \tag{11}$$

Where $k$ is a positive constant gain. It will be shown that the observer in (11) fulfil the estimation. Let:

$$e(t) = b - \hat{b} \tag{12}$$

Assuming $\hat{b}$ as an estimation of the unknown parameter $b$. Then:

$$\dot{e}(t) = \dot{b} - \dot{\hat{b}} = -\dot{\hat{b}} \tag{13}$$

Candidate the Lyapunov functions $V$ as:

$$V = \frac{1}{2}e^2 \tag{14}$$

The first derivative of Lyapunov function is:

$$\dot{V} = e\dot{e} = (b - \hat{b})(-\dot{\hat{b}}) \tag{15}$$

According to the third equation of system (7), we have:

$$\dot{V} = (\frac{xy - D^{q_3}z}{z} - \hat{b})(-\dot{\hat{b}}) \tag{16}$$

Substitution of the observer adaptation law in equation (11) into Equation (16) achieves the following derivative of the Lyapunov function:

$$\dot{V} = -kz\left[\hat{b}z - (xy - D^{q_3}z)\right]^2 < 0 \tag{17}$$

Since negative sign of the first derivative of the Lyapunov function is satisfied; $\hat{b}(t)$ finally converges to the real parameter $b$. It is now of an aim to proof that $\hat{b}(t)$ converges to $b$ with an exponential rate. From Equations (11) and (13), it is obtained:

$$\dot{e} = kz\left[\hat{b}z - (xy - D^{q_3}z)\right] \tag{18}$$

By the third equation of system (7), we have:

$$D^{q_3}z = xy - bz \tag{19}$$

By substitution of (19) in (18), the following equation is achieved:

$$e(t) + kz^2e(t) = 0 \tag{20}$$

For $k > 0$ immediately follows that $e(t)$ converges to zero. Therefore, $\hat{b}(t)$ converges to $b$ with an exponential rate when $t \to \infty$. This means, the identification of the parameter of the system $\alpha = 3b - 8$ is achieved. The functionality of the proposed observer is shown in Figure 3 for different values of unknown parameter $\alpha$ that is $(\alpha = 0, 0.8, 1)$. Figure 3 shows the quality of the designated observer in Equation (11) when the estimated parameter approaches the exact value.

## The feedback control

In this section, we primarily assumed that the parameter of the unified system has already been identified with the prescribed method discussed earlier on. As a consequence of the certainty equivalence principle, the goal is to design a controller to stabilize the system at their unstable equilibrium points, after the unknown parameter is off-line identified. Suppose that the identified parameter in section 3 is $\alpha = 1$ which means the Chen unified system is being dealt, the equilibrium point will be of the target to get stabilized in both cases of; commensurate and incommensurate fractional order. The performance of the proposed controller will be compared with that of the classical state feedback controller.

## Classical state feedback control technique

For the classical feedback control, state variables of the system are often multiplied by a coefficient as a feedback gain to be added in the right-hand side to construct a control effort. The controlled fractional order Chen system is accordingly given by:

$$\begin{cases} \dfrac{d^{q_1}x}{dt^{q_1}} = 35(y - x) \\ \dfrac{d^{q_2}y}{dt^{q_2}} = -7x - xz + 28y - u \\ \dfrac{d^{q_3}z}{dt^{q_3}} = xy - 3z \end{cases} \tag{21}$$

Where $u$ is an external control input which derives the trajectory of Equation (7) towards the unstable equilibrium point. The control effort consists of a single variable state feedback control law of the following form:

$$u = k(y - \bar{y}) \tag{22}$$

Where; $\bar{y}$ is the equilibrium point of the second state variable and $k$ is the feedback gain. Substituting Equation (22) into (21) yields:

$$\begin{cases} \dfrac{d^{q_1}x}{dt^{q_1}} = 35(y - x) \\ \dfrac{d^{q_2}y}{dt^{q_2}} = -7x - xz + 28y - k(y - \bar{y}) \\ \dfrac{d^{q_3}z}{dt^{q_3}} = xy - 3z \end{cases} \tag{23}$$

### *Stabilizing the origin equilibrium point*

Replacement of the coordinate of the origin $O$ into Equation (23) yields:

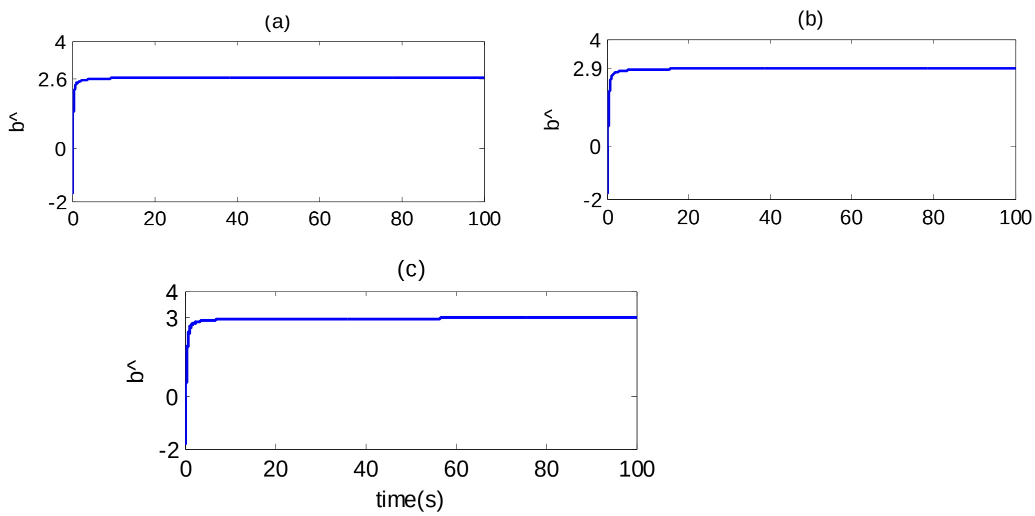


**Figure 3.** Identification of the unknown parameter of the fractional order unified chaotic system via designated observer in Eq. (11) for (a) $\alpha=0$ (b) $\alpha=0.8$ (c) $\alpha=1$.

$$\begin{cases} \dfrac{d^{q_1}x}{dt^{q_1}} = 35(y-x) \\ \dfrac{d^{q_2}y}{dt^{q_2}} = -7x - xz + 28y - ky \\ \dfrac{d^{q_3}z}{dt^{q_3}} = xy - 3z \end{cases} \tag{24}$$

The Jacobian matrix of system (24) is therefore obtained by:

$$J = \begin{bmatrix} -35 & 35 & 0 \\ -7 & 28-k & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

The characteristic polynomial of the Jacobian matrix is accordingly found as:

$$P(\lambda) = \lambda^3 + \lambda^2(k+10) + \lambda(38k-714) + 105k - 2205 \tag{25}$$

While the discriminant of the system is shown in Equation (6). It immediately follows that:

$$\begin{cases} a_1 = k+10 \\ a_2 = 38k - 714 \\ a_3 = 105k - 2205 \end{cases} \tag{26}$$

The selection $23.3438 < k < 31.695$ or $k > 94.305$ causes $D(P) > 0, a_1 > 0, a_2 > 0, a_3 > 0, a_1a_2 > a_3$; this, according to proposition 1, emphasizes that all of the eigenvalues are found real and negative. Consequently, the Routh-Hurwitz criterion meets the necessary and sufficient conditions in Theorem1.

***Stabilizing the unstable equilibrium point*** $C^{\pm}$

The case $\alpha=1$ forms the $C^{\pm}$ coordinates in (8) to:

$$C^{\pm} = (\pm 7.94, \pm 7.94, 21) \tag{27}$$

Substitution of the $C^{+}$ coordinate into Equation (23) provides:

$$\begin{cases} \dfrac{d^{q_1}x}{dt^{q_1}} = 35(y-x) \\ \dfrac{d^{q_2}y}{dt^{q_2}} = -7x - xz + 28y - k(y-7.94) \\ \dfrac{d^{q_3}z}{dt^{q_3}} = xy - 3z \end{cases} \tag{28}$$

The relevant Jacobian matrix of system (28) is found by:

$$J = \begin{bmatrix} -35 & 35 & 0 \\ -28 & 28-k & -7.94 \\ 7.94 & 7.94 & -3 \end{bmatrix}$$

Accordingly, the corresponding characteristic polynomial is found as:

$$P(\lambda) = \lambda^3 + \lambda^2(k+10) + \lambda(38k+84) + 105k + 4410 \tag{29}$$

This provides the coefficients of the Routh-Hurwitz as:

$$\begin{cases} a_1 = k+10 \\ a_2 = 38k+84 \\ a_3 = 105k+4410 \end{cases} \tag{30}$$

When $k > 125.48$, we have $D(P) > 0, a_1 > 0, a_2 > 0, a_3 > 0, a_1 a_2 > a_3$, and from proposition 1 all of the eigenvalues are also found real and negative. This means the trajectory of the controlled fractional order of system (24) is asymptotically stabilized at the equilibrium point $C^+$. The same procedure asymptotically stabilizes the equilibrium point $C^-$. In addition for $k = 6.0587$, we have $D(P) < 0, a_1 > 0, a_2 > 0, a_1 a_2 = a_3$, which means the real parts of the complex conjugate eigenvalues are zero, namely for any $Q \in [0,1)$, all the eigenvalues satisfy theorem 1. Therefore, the trajectory of the controlled fractional order of system (24) is asymptotically stabilized at the equilibrium point $C^{\pm}$.

### *Enhancing the feedback control scheme, via multi control inputs*

It is less likely to control a complex system by only one variable state feedback or might be of large value. Thus, multi variables state feedback is of interest as a feedback gain. This method is so called "enhancing the feedback control" (Zhu, 2009). The control method will be expressed in the following:
Consider the dynamics:

$$\begin{cases} \dfrac{d^{q_1} x}{dt^{q_1}} = 35(y-x) - u_1 \\ \dfrac{d^{q_2} y}{dt^{q_2}} = -7x - xz + 28y - u_2 \\ \dfrac{d^{q_3} z}{dt^{q_3}} = xy - 3z - u_3 \end{cases} \tag{31}$$

Where $u_1, u_2, u_3$ are the external control inputs. These are designed so as to derive the trajectory of Eq. (7) towards the unstable equilibrium point. In fact, the control laws are defined as follows:

$$\begin{aligned} u_1 &= k_1(x-\bar{x}) \\ u_2 &= k_2(y-\bar{y}) \\ u_3 &= k_3(z-\bar{z}) \end{aligned} \tag{32}$$

Where the coordinate $(\bar{x},\bar{y},\bar{z})$ is the desired unstable equilibrium point of the chaotic Equation. (7), and $k_1, k_2, k_3$ are the feedback gains. Substituting Equation (32) into (31) provides the dynamic as:

$$\begin{cases} \dfrac{d^{q_1} x}{dt^{q_1}} = 35(y-x) - k_1(x-\bar{x}) \\ \dfrac{d^{q_2} y}{dt^{q_2}} = -7x - xz + 28y - k_2(y-\bar{y}) \\ \dfrac{d^{q_3} z}{dt^{q_3}} = xy - 3z - k_3(z-\bar{z}) \end{cases} \tag{33}$$

### *Stabilizing the origin equilibrium point, $O$*

Substituting the coordinate of the origin, $O$ into Eq. (33), yields:

$$\begin{cases} \dfrac{d^{q_1} x}{dt^{q_1}} = 35(y-x) - k_1 x \\ \dfrac{d^{q_2} y}{dt^{q_2}} = -7x - xz + 28y - k_2 y \\ \dfrac{d^{q_3} z}{dt^{q_3}} = xy - 3z - k_3 z \end{cases} \tag{34}$$

The Jacobian matrix of system (34) is accordingly found by:

$$J = \begin{bmatrix} -35-k_1 & 35 & 0 \\ -7 & 28-k_2 & 0 \\ 0 & 0 & -3-k_3 \end{bmatrix}$$

This establishes the corresponding characteristic equation as:

$$\begin{aligned} P(\lambda) = {} & \lambda^3 + \lambda^2(k_1+k_2+k_3+10) \\ & + \lambda(-714+38k_2+7k_3-25k_1+k_1k_2+k_1k_3+k_2k_3) \\ & -2205+105k_2-735k_3+35k_2k_3-84k_1-28k_1k_3+3k_1k_2+k_1k_2k_3 \end{aligned} \tag{35}$$

This obtains:

$$\begin{cases} a_1 = k_1+k_2+k_3+10 \\ a_2 = -714+38k_2+7k_3-25k_1+k_1k_2+k_1k_3+k_2k_3 \\ a_3 = -2205+105k_2-735k_3+35k_2k_3-84k_1-28k_1k_3+3k_1k_2+k_1k_2k_3 \end{cases} \tag{36}$$

Choosing $k_1 = -3, k_2 = 25$ and $-2 \le k_3 < 14.2345$, results $D(P) > 0, a_1 > 0, a_2 > 0, a_3 > 0, a_1 a_2 > a_3$. These again mean all eigenvalues are real and negative. Therefore the trajectory of the controlled fractional order of system (33) is asymptotically stable at the equilibrium point $O$. When $k_1 = -k_3 - 10,\ k_3 = -3 + k_2$ and $12.3475 < k_2 < 43.6525$, we have $D(P) < 0, a_1 > 0, a_2 > 0, a_1 a_2 = a_3$, then from proposition 2 $\forall Q \in [0,1)$ the controlled fractional order of system (33) is asymptotically stable.

### *Stabilizing the unstable equilibrium point $C^{\pm}$*

From Equation (27), by substituting the coordinate of $C^{+}$ into Equation (33), yields the Jacobian matrix as:

$$J = \begin{bmatrix} -35-k_1 & 35 & 0 \\ -28 & 28-k_2 & -7.94 \\ 7.94 & 7.94 & -3-k_3 \end{bmatrix}$$

The characteristic equation of the Jacobian matrix is accordingly obtained by:

$$\begin{aligned} P(\lambda) &= \lambda^3 + \lambda^2(k_1+k_2+k_3+10) \\ &+ \lambda(84+38k_2+7k_3-25k_1+k_1k_2+k_1k_3+k_2k_3) \\ &+ 4413+105k_2-21k_1+35k_2k_3-28k_1k_3+3k_1k_2+k_1k_2k_3 \end{aligned} \quad (37)$$

This immediately follows that:

$$\begin{cases} a_1 = k_1+k_2+k_3+10 \\ a_2 = 84+38k_2+7k_3-25k_1+k_1k_2+k_1k_3+k_2k_3 \\ a_3 = 4413+105k_2+35k_2k_3-21k_1-28k_1k_3+3k_1k_2+k_1k_2k_3 \end{cases} \quad (38)$$

When $k_1 = 3,\ k_2 = 1$ and $k_3 > 58.3$ we have $D(P) > 0, a_1 > 0, a_2 > 0, a_3 > 0, a_1a_2 > a_3$. This verifies that all of the eigenvalues are real and located in the left half of the s-plane. Then the Routh-Hurwitz conditions are the necessary and sufficient conditions for the existence of Eq. 3. The selection of $k_1 = -k_3 - 10, k_2 = 31.5238$ and $k_3 = -3 + k_2$ leads to $D(P) < 0, a_1 > 0, a_2 > 0, a_1a_2 = a_3$. Hence, from proposition 2, the real part of the complex conjugate eigenvalues are zero and the controlled fractional order Chen system (33) is asymptotically stable at the equilibrium point $C^{\pm}$.

## SIMULATION RESULT

A simulation procedure has been carried out using MATLAB SIMULINK in two individual cases of the commensurate and incommensurate order. Dormand-Prince solver is used during the numerical simulation. The order of the fractional system is set to $q = 0.98$ for commensurate order and $(q_1, q_2, q_3) = (0.85, 0.95, 0.9)$ for incommensurate type. This setting is chosen to ensure the occurrence of the chaos in absence of the control effort. Initial conditions of states are also selected as $(x(0), y(0), z(0)) = (15, 10, 6)$ for the same reason.

### Simulation of the classical feedback control

1. System (23) stabilizes the unstable equilibrium point $O = (0,0,0)$, as depicted in Figure 4. The selection $k = 25$ results $D(P) > 0, a_1 > 0, a_2 > 0, a_3 > 0, a_1a_2 > a_3$ and assigns the three roots of polynomial equation at: $\lambda_1 = -3, \lambda_2 = -5.23, \lambda_3 = -26.77$ which showing system (23) is asymptotically stable at $O = (0,0,0)$.
2. The simulation is fulfilled for the unstable equilibrium points $C^{\pm} = (\pm 7.94, \pm 7.94, 21)$ using a classical feedback method. $k = 127$, makes $D(P) > 0$ and $a_1 > 0, a_2 > 0, a_3 > 0, a_1a_2 > a_3$ which causes the three roots of polynomial equation to be $\lambda_1 = -4.0604, \lambda_2 = -73.4, \lambda_3 = -59.54$ for $C^{\pm} = (\pm 7.94, \pm 7.94, 21)$. This verifies that system (23) is asymptotically stable at $C^{\pm}$. Figure 5 confirms the stabilization of the equilibrium point $C^{+}$ whereas Figure 6 shows the stabilization of the equilibrium point $C^{-}$. The simulation results are shown in Figures 5a and 6a, for the commensurate order whereas, Figures 5b and 6b, show the case for the incommensurate order of fractional system.
3. Similarly, the selection $k = 6.0587$ causes $D(P) < 0, a_1 > 0, a_2 > 0, a_1a_2 = a_3$ which forces the roots of $P(\lambda) = 0$ are $\lambda_1 = -31.5223, \lambda_2 = 32.1026j, \lambda_3 = -32.1026j$. This confirms system (21) is asymptotically stable at $C^{\pm}$ for any $Q \in [0,1)$. Likewise Figures 7 and 8 prove the stabilization of the equilibrium points $C^{+}$ and $C^{-}$ respectively.

### Simulation of the enhancing feedback control

The simulation is similarly carried out for the multi input control case to stabilize the unstable equilibrium points in two individual cases of the commensurate and incommensurate order respectively.

- The results for the stabilization of the origin equilibrium point $O = (0,0,0)$, are shown in Figures 9 and 10. While Figures 9a and 10a show for commensurate and Figures 9b and 10b for incommensurate fractional order system. The selection $k_1 = -3, k_2 = 25, k_3 = -2$ provides $D(P) > 0$, and $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_1a_2 > a_3$ which make the three roots of polynomial equation to be: $\lambda_1 = -1$, $\lambda_2 = -6.67$, $\lambda_3 = -22.32$. This confirms that

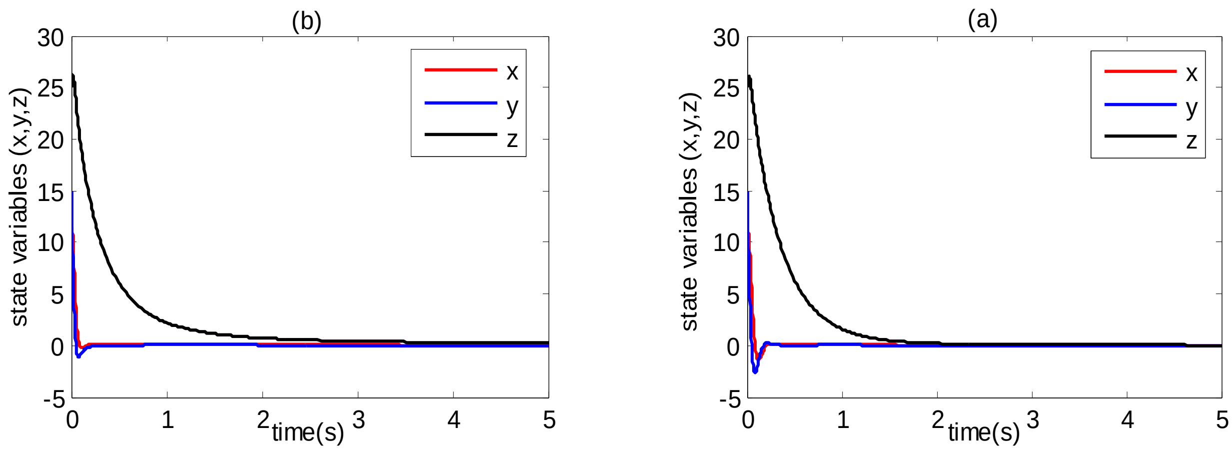


**Figure 4.** Time response of the $x, y, z$ states of the controlled Equation (23) during the stabilization of the unstable equilibrium point $O=(0,0,0)$ for $D(P)>0$ (a) with commensurate order $q=0.98$ (b) with incommensurate order $(q_1, q_2, q_3)=(0.85, 0.95, 0.9)$

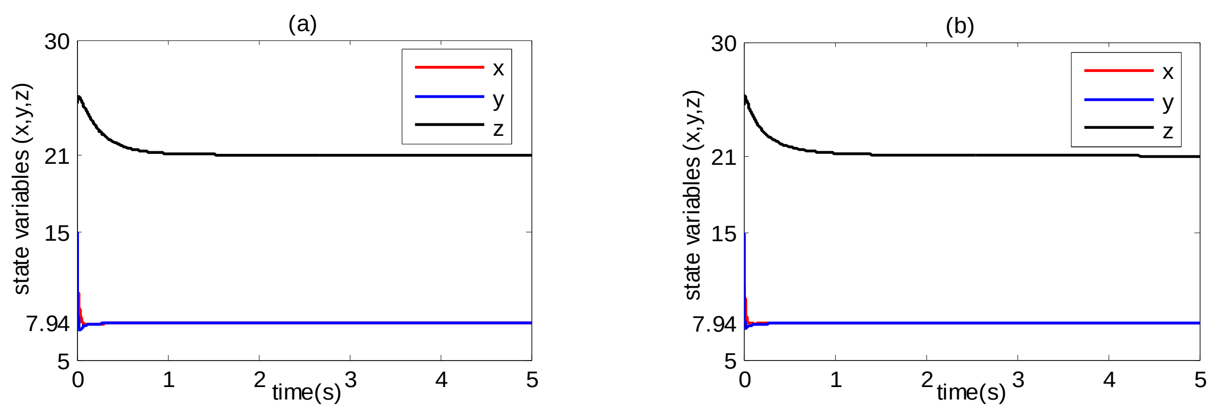


**Figure 5.** Time response of the $x, y, z$ states of the controlled Equation (23) during the stabilization of the unstable equilibrium point $C^{+}=(7.94, 7.94, 21)$ for $D(P)>0$ (a) with commensurate order $q=0.98$ (b) with incommensurate order $(q_1, q_2, q_3)=(0.85, 0.95, 0.9)$

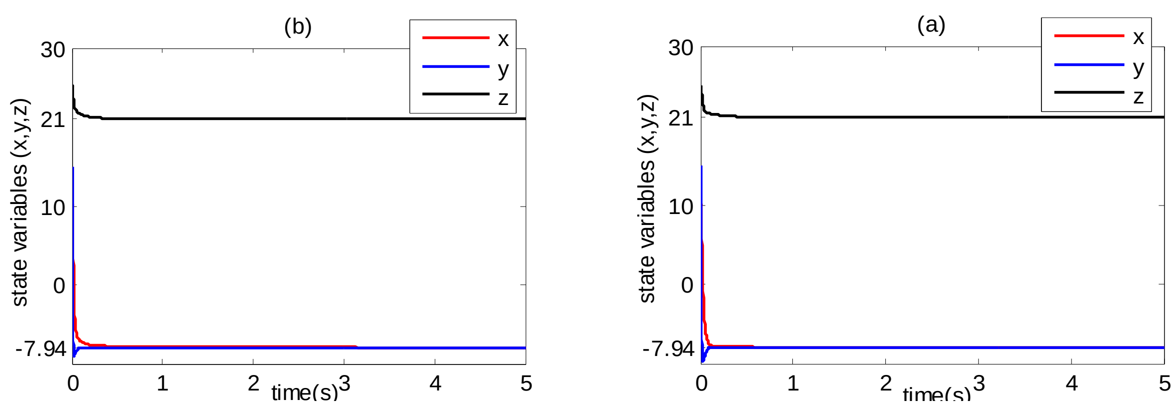


**Figure 6.** Time response of the $x, y, z$ states of the controlled Equation (23) during the stabilization of the unstable equilibrium point $C^{-}=(-7.94, -7.94, 21)$ for $D(P)>0$ (a) with commensurate order $q=0.98$ (b) with incommensurate order $(q_1, q_2, q_3)=(0.85, 0.95, 0.9)$.

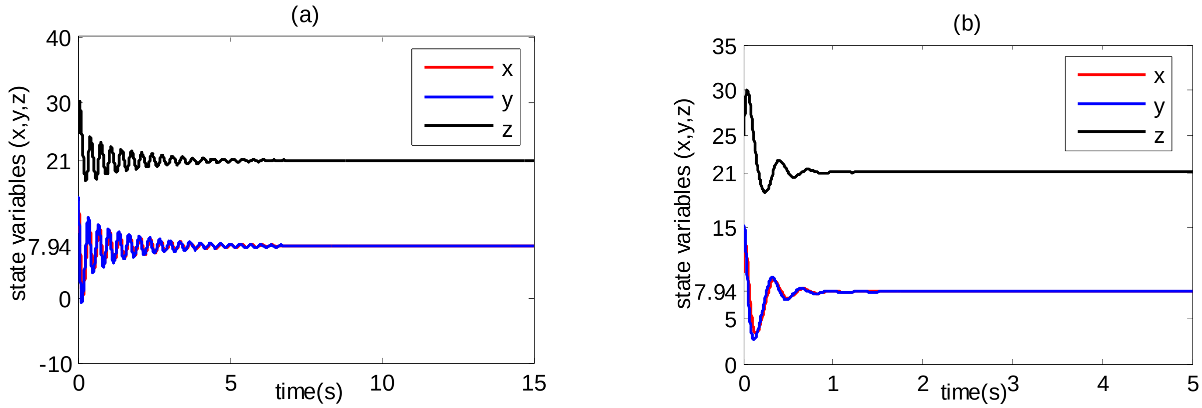


**Figure 7.** Time response of the $x, y, z$ states of the controlled Equation (23) during the stabilization of the unstable equilibrium point $C^{+} = (7.94, 7.94, 21)$ for $D(P) < 0$ (a) with commensurate order $q = 0.98$ (b) with incommensurate order $(q_1, q_2, q_3) = (0.85, 0.95, 0.9)$.

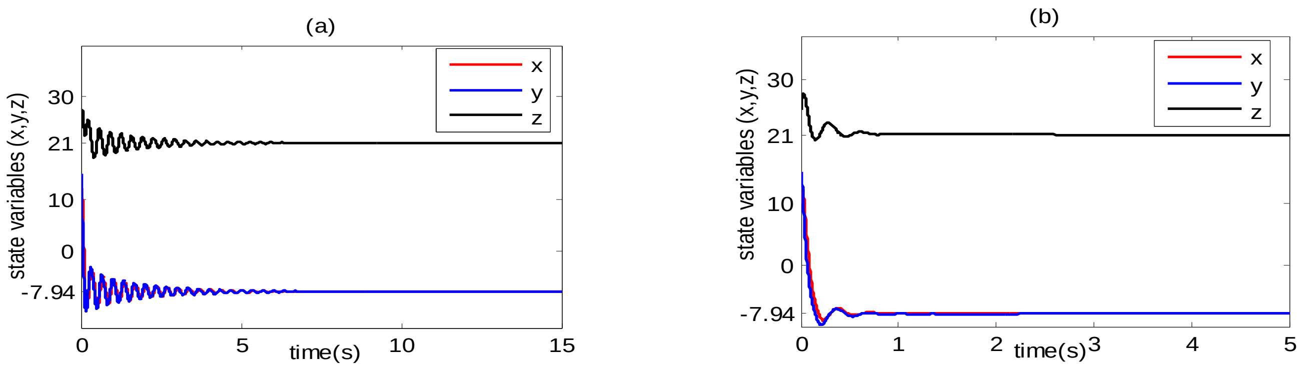


**Figure 8.** Time responses of the $x, y, z$ states of the controlled Equation (23) during the stabilization of the unstable equilibrium point $C^{-} = (-7.94, -7.94, 21)$ for $D(P) < 0$ (a) with commensurate order $q = 0.98$ (b) with incommensurate order $(q_1, q_2, q_3) = (0.85, 0.95, 0.9)$.

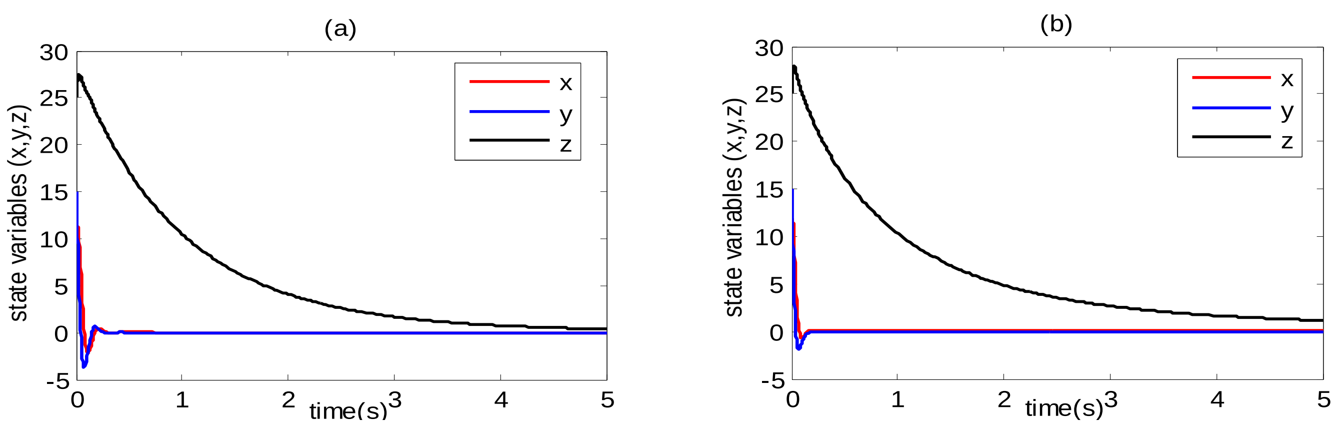


**Figure 9.** Time response of the $x, y, z$ states of the controlled Equation (33) during the stabilization of the unstable equilibrium point $O = (0,0,0)$ for $D(P) > 0$ (a) with commensurate order $q = 0.98$ (b) with incommensurate order $(q_1, q_2, q_3) = (0.85, 0.95, 0.9)$.

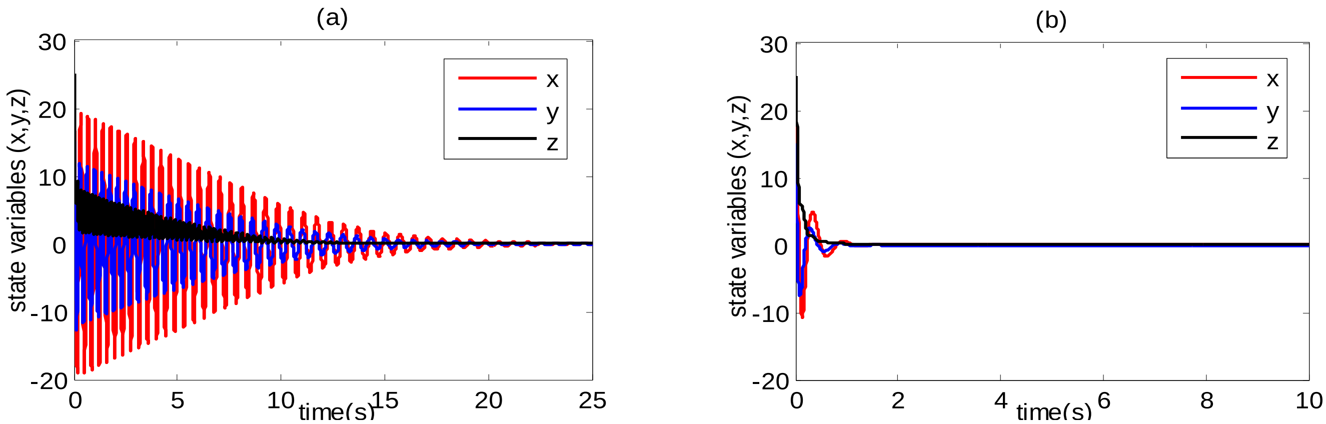


**Figure 10**. Time response of the $x, y, z$ states of the controlled Equation (33) during the stabilization of the unstable equilibrium point $O = (0,0,0)$ for $D(P) < 0$ (a) with commensurate order $q = 0.98$ (b) with incommensurate order $(q_1, q_2, q_3) = (0.85, 0.95, 0.9)$

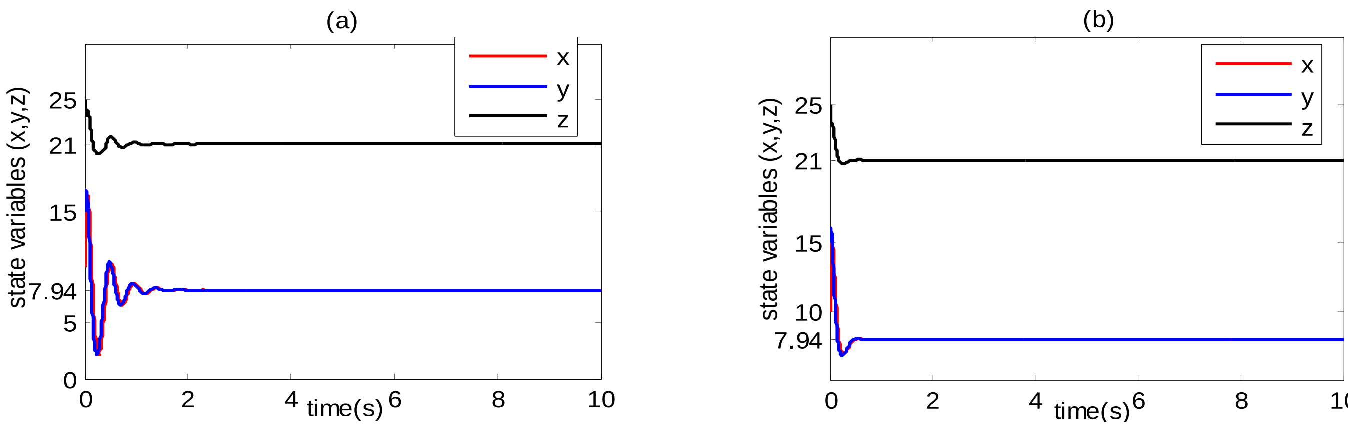


**Figure 11.** Time responses the $x, y, z$ states of the controlled Equation (33) during the stabilization of the unstable equilibrium point $C^+ = (7.94, 7.94, 21)$ for $D(P) > 0$ (a) with commensurate order $q = 0.98$ (b) with incommensurate order . $(q_1, q_2, q_3) = (0.85, 0.95, 0.9)$ .

system (33) is asymptotically stable at $O = (0,0,0)$. Furthermore $k_1 = -22$, $k_2 = 15$, $k_3 = 12$ cause $D(P) < 0, a_1 > 0, a_2 > 0$, $a_1 a_2 = a_3$. This again makes the three roots of polynomial equation are: $\lambda_1 = -15$, $\lambda_2 = 8.72j$, $\lambda_3 = -8.72j$ which means system (33) is asymptotically stable at $O = (0,0,0)$ for any $Q \in [0,1)$.

- The simulation is continued to stabilize the equilibrium points $C^{\pm} = (\pm 7.94, \pm 7.94, 21)$ of system (33) using the enhancing feedback method. The selection $k_1 = -3, k_2 = 1, k_3 = 60$, provides $D(P) > 0$, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_1 a_2 > a_3$. This accordingly produces the three roots of polynomial equation to become $\lambda_1 = -3.9, \lambda_2 = -63.2, \lambda_3 = -6.9$, for $C^{\pm} = (\pm 7.94, \pm 7.94, 21)$. This also confirms the asymptotical stability at $C^{\pm}$ for the system in (33). Figures 11 and 12 show the stabilization of the equilibrium points $C^+$ and $C^-$ respectively.

- Similarly, $k_1 = -38.5238, k_2 = 31.5238, k_3 = 28.5238$ produce $D(P) < 0, a_1 > 0, a_2 > 0$, $a_1 a_2 = a_3$. Therefore, the roots of $P(\lambda) = 0$ are $\lambda_1 = -6.0706, \lambda_2 = 19.669j, \lambda_3 = -19.669j$, which show the asymptotical stability of system (33) at $C^{\pm}$ for any $Q \in [0,1)$. Figures 13 and 14 show the stabilization of the equilibrium points $C^+$ and $C^-$ respectively.

The proposed control technique is found comparable with respect to a Lyapunov based stabilization method

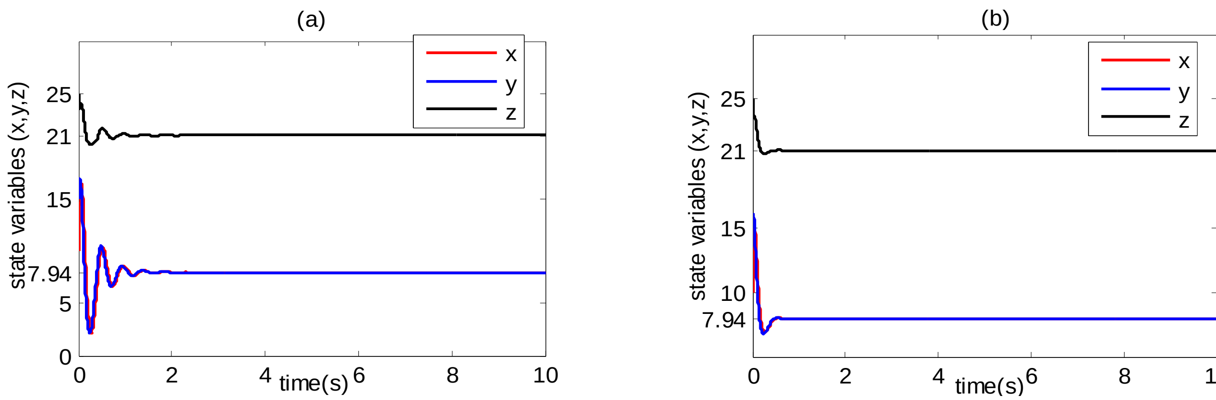


**Figure 11.** Time responses the $x, y, z$ states of the controlled Equation (33) during the stabilization of the unstable equilibrium point $C^+ = (7.94, 7.94, 21)$ for $D(P) > 0$ (a) with commensurate order $q = 0.98$ (b) with incommensurate order . $(q_1, q_2, q_3) = (0.85, 0.95, 0.9)$.

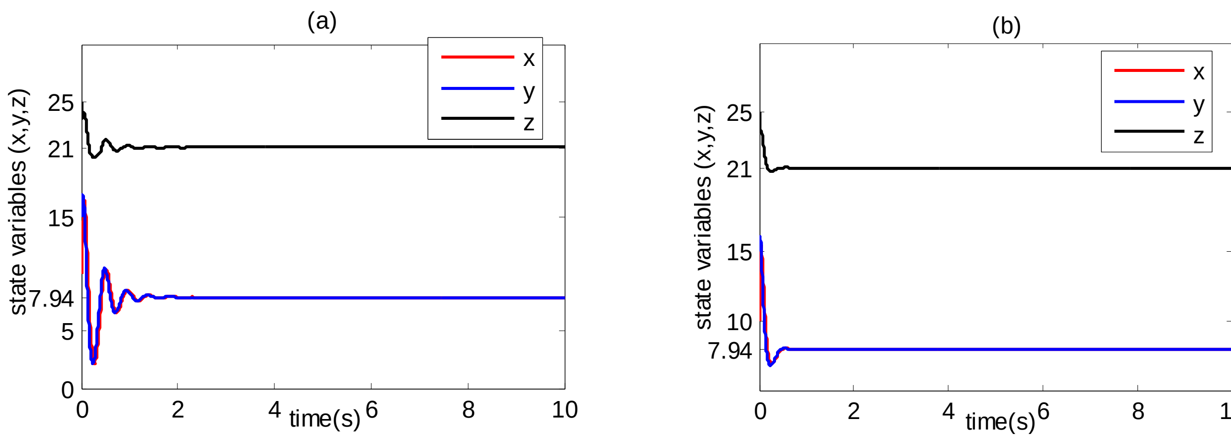


**Figure 12.** Time responses for the $x, y, z$ states of the controlled Equation (33) during the stabilization of the unstable equilibrium point $C^- = (-7.94, -7.94, 21)$ for $D(P) > 0$ (a) with commensurate order $q = 0.98$ (b) with incommensurate order $(q_1, q_2, q_3) = (0.85, 0.95, 0.9)$

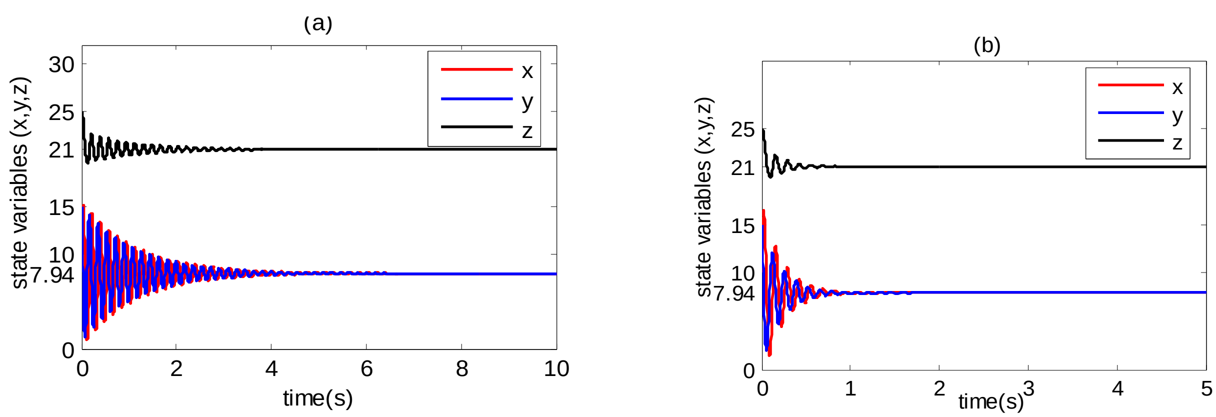


**Figure 13.** Time response of the $x, y, z$ states of the controlled Equation (33) during the stabilization of the unstable equilibrium point $C^+ = (7.94, 7.94, 21)$ for $D(P) < 0$ (a) with commensurate order $q = 0.98$ (b) with incommensurate order $(q_1, q_2, q_3) = (0.85, 0.95, 0.9)$.

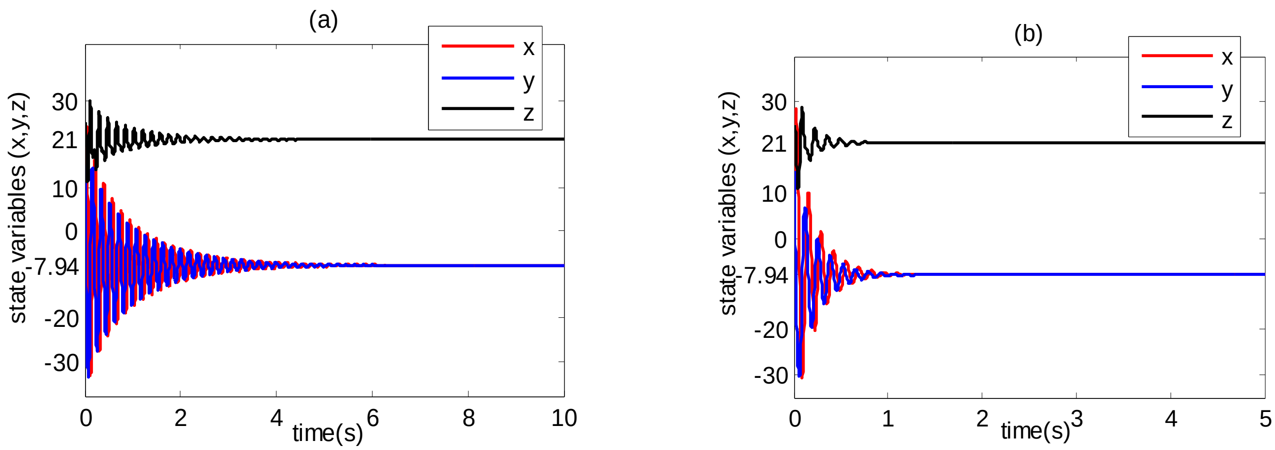


**Figure 14.** Time response of the $x, y, z$ states of the controlled Equation (33) during the stabilization of the unstable equilibrium point $C^{-} = (-7.94, -7.94, 21)$ for $D(P) < 0$ (a) with commensurate order $q = 0.98$ (b) with incommensurate order $(q_1, q_2, q_3) = (0.85, 0.95, 0.9)$.

(Ahmad et al., 2004). The Lyapunov theory is used (Ahmad et al., 2004) to prove the stability of the system by a state feedback controller only at (0, 0, 0) equilibrium point. However, the state feedback gain was found less likely to become small (Ahmad et al., 2004). The stabilization was also achieved with price of losing the stability speed in some cases.

## Conclusion

This paper deals with the classical feedback and the enhancing feedback control methods to reach the stabilization of unstable equilibrium points in the fractional order Chen system. On the basis of the Routh-Hurwitz and the relevant stability theorems for fractional-order systems, the authors get the sufficient conditions for achieving stabilization of unstable equilibrium points in the fractional order Chen system with commensurate and incommensurate order theoretically. Simulation results show the effectiveness of the control methods that can stabilize the chaotic trajectory of the fractional order Chen system with commensurate and incommensurate order to unstable equilibrium points.

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