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On meromorphic functions with fixed residue defined using convolution

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In this paper, we introduced a new subclass of meromorphic functions with residue $\xi = \mathrm{Res}(f, w)$, which is defined by means of a Hadamard product (or convolution). Then we determine some properties such as coefficient bound, distortion theorems, radius of starlikeness and convexity for this class.

Key words: Analytic, meromorphic functions, Hadamard product, convolution

INTRODUCTION

Let $H(U)$ be the set of functions which are regular in the unit disc $U = \{z : |z| < 1\}$, $A = \{f \in H(U) : f(0) = f'(0) - 1 = 0\}$ and S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1)$$

that are analytic and univalent in unit disc U.

Denoted by $S^*(\alpha)$ and $C(\alpha)$, $(0 \le \alpha < 1)$,the subclasses of functions in S that are starlike of order α and convex of order α, respectively. Analytically, $f \in S^*(\alpha)$ if and only if f is of the form (1) and satisfies

$$\mathrm{Re}\left\{\frac{z f'(z)}{f(z)}\right\} > \alpha \ , \quad z \in U \cdot \qquad (2)$$

Similarly, $f \in C(\alpha)$ if and only if f is of the form (1) and satisfies

$$\mathrm{Re}\left\{1 + \frac{z f''(z)}{f'(z)}\right\} > \alpha \ , \quad z \in U \qquad (3)$$

Let T denote the class of functions analytic in unit disc U that are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (4)$$

and let $T^*(\alpha) = T \cap S^*(\alpha)$ and $K(\alpha) = T \cap C(\alpha)$. The class $T^*(\alpha)$ and allied classes posses some interesting properties and have been studied by Silverman (1975, 1976) and Silvia (1979) and others.

Gupta and Jain (1976) extended some of the results of Silverman to functions of the form (4) that are starlike of order α and type β, $(0 < \beta \le 1)$. The class of starlike

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functions of order α and type β was introduced by Junega and Mogra (1975) who also made a detailed study about it (Junega and Mogra, 1978; Mogra and Junega, 1977).

Let w be a fixed point in U and $A(w)=\{ f \in H(U): f(w)=f'(w)-1=0 \}$. It is easy to see that a function $f \in A(w)$ has the series expansion:

$$f(z)=(z-w)+a_2(z-w)^2+\dots .$$

Kanas and Ronning (1999) introduced the following classes:

$$S(w)=\{ f \in A(w): f \textit{ is univalent in } U \}$$

$$ST(w)=\left\{ f \in S(w): \operatorname{Re}\left(\frac{(z-w)f'(z)}{f(z)} \right) > 0 \, , \, z \in U \right\}$$

$$CV(w)=\left\{ f \in S(w): 1+\operatorname{Re}\left(\frac{(z-w)f''(z)}{f'(z)} \right) > 0 \, , \, z \in U \right\}.$$

The class $ST(w)$ is defined by the geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$ and the corresponding class $CV(w)$ is defined by the property that the image of any circular arc centered w is convex.

It is obvious that there exists a natural " Alexander relation " between the classes $ST(w)$ and $CV(w)$:

$$f \in CV(w) \;\; \textit{iff} \;\; (z-w)f' \in ST(w).$$

Let $P(w)$ denote the class of all functions

$$p(z)=1+\sum_{n=1}^{\infty} B_n (z-w)^n \tag{5}$$

that are regular in U and satisfy $p(w)=1$ and $\operatorname{Re} p(z)>0$ for $z \in U$.

Let $\sum$ denote the class of the functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty} a_n z^n \tag{6}$$

that are regular and univalent in $U^*=U-\{0\}$ with a simple pole at the origin with residue 1.

We denote this class by $\sum$. Let $\sum_S, \sum^*(\alpha)$ and $\sum_C(\alpha)$, $(0 \le \alpha < 1)$, denote the subclasses of $\sum$ that are univalent, meromorphically starlike of order α and meromorphically convex of order α, respectively. We say that a function $f \in \sum$ is meromorphically starlike of order α and belongs to the class $\sum^*(\alpha)$ if it satisfies the inequality

$$-\operatorname{Re}\left\{ \frac{z f'(z)}{f(z)} \right\} > \alpha \;\; , \quad z \in U^* . \tag{7}$$

Similarly, a function $f \in \sum$ is meromorphically convex of order α and belongs to the class $\sum_C(\alpha)$ if it satisfies the inequality

$$-\operatorname{Re}\left\{ 1+\frac{z f''(z)}{f'(z)} \right\} > \alpha \;\; , \quad z \in U^* . \tag{8}$$

The class $\sum^*(\alpha)$ and other similar classes have been extensively studied by Pommerenke (1963), Clunie (1959), Miller (1970), Royster (1963), and others.

Mogra et al. (1985) defined the class of meromorphically starlike functions of order α type β as follows:

A function $f \in \sum$ is said to be meromorphically starlike functions of order α type β if it satisfies the condition

$$\left| z\frac{f'(z)}{f(z)}+1 \right| < \beta \left| z\frac{f'(z)}{f(z)}+2\alpha-1 \right| \tag{9}$$

for some α, β $(0 \le \alpha < 1, \, 0 < \beta \le 1)$ and for $z \in U^*$.

For $0 \le w < 1$, let $\sum_w$ denote the class of functions f which are meromorphic and univalent in the unit disc U with the normalization $\lim_{z \to w} f(z) = \infty$.

Let A_w denote the set of function analytic in $U-\{w\}$ with the topology given by uniform convergence on compact subsets of $U-\{w\}$. Then A_w is locally convex linear topological space and $\sum_w$ is a compact subset of A_w (Schober, 1975).

In the annulus $\{z : w < |z| < 1\}$ every function f in $\sum_w$ has an expansion of the form

$$f(z) = \frac{\xi}{z-w} + \sum_{n=1}^{\infty} c_n z^n \qquad (10)$$

where $\xi = \mathrm{Res}(f, w)$ with $\xi \in \mathbb{C} - \{0\}$, $z \in U - \{w\}$ or we may set for $U_w = \{z : 0 < |z-w| < 1-w\}$

$$f(z) = \frac{\xi}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n . \qquad (11)$$

A function f in $\sum_w$ is said to be meromorphically starlike of order α $(0 \le \alpha < 1)$ if and only if

$$\mathrm{Re}\left\{-\frac{(z-w)f'(z)}{f(z)}\right\} > \alpha \;, \quad (z-w) \in U . \qquad (12)$$

We denote by $\sum_w^*(\alpha)$ the class of all meromorphic starlike functions of order α.

Similarly, a function f in $\sum_w$ is said to be meromorphically convex of order α $(0 \le \alpha < 1)$ if and only if

$$\mathrm{Re}\left\{-1-\frac{(z-w)f''(z)}{f'(z)}\right\} > \alpha \;, \quad (z-w) \in U . \qquad (13)$$

We denote by $\sum_{C_w}(\alpha)$ the class of all meromorphic convex functions of order α.

If $f(z) = \frac{\xi}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n$ and $g(z) = \frac{\sigma}{z-w} + \sum_{n=1}^{\infty} b_n (z-w)^n$ are analytic in U^*, then their Hadamard product (or convolution) is defined by

$$(f * g)(z) = \frac{\xi\sigma}{z-w} + \sum_{n=1}^{\infty} a_n b_n (z-w)^n . \qquad (14)$$

For the function $f(z) = \frac{\xi}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n$ in the class $\sum_w$, we define the following differential operator:

$$I_{t,\xi}^0 f(z) = f(z)$$

$$I_{t,\xi}^1 f(z) = (1-t) f(z) + t(z-w) f'(z) + \frac{2t\xi}{z-w},$$

and for $m = 1, 2, 3, \ldots$ we can write:

$$I_{t,\xi}^m f(z) = (1-t)\left(I_{t,\xi}^{m-1} f(z)\right) + t(z-w)\left(I_{t,\xi}^{m-1} f(z)\right)' + \frac{2t\xi}{z-w}$$

$$= \frac{\xi}{z-w} + \sum_{n=1}^{\infty} [1+(n-1)t]^m a_n (z-w)^n$$

where $t \ge 0$ *and* $z-w \in U$.

In this study, we introduce a new class $\sum_w^*(\phi, \varphi, \alpha, \beta, k)$ of meromorphic functions of the form (11) with the help of the differential operator $I_{t,\xi}^m$ and convolution as follows:

Suppose the functions $\phi(z)$ and $\varphi(z)$ are given by

$$\phi(z) = \frac{c}{z-w} + \sum_{n=1}^{\infty} \lambda_n (z-w)^n \quad , \; c \in \mathbb{C} - \{0\}$$

and

$$\varphi(z) = \frac{d}{z-w} + \sum_{n=1}^{\infty} \mu_n (z-w)^n \;, d \in \mathbb{C} - \{0\}.$$

Then we say that $f \in \sum_w$ is in the class $\sum_w^*(\phi, \varphi, \alpha, \beta, k)$ if

$$\left|\frac{I_{t,\xi}^m (f * \phi)}{I_{t,\xi}^m (f * \varphi)} + k\right| < \beta \left|\frac{I_{t,\xi}^m (f * \phi)}{I_{t,\xi}^m (f * \varphi)} + 2\alpha - k\right|$$

for some α, β $(0 \le \alpha < 1 \,,\; 0 < \beta \le 1)$, $t \ge 0$, $\xi \in \mathbb{C} - \{0\}$, $0 \le k \le 1$, provided that $I_{t,\xi}^m (f * \varphi) \ne 0$, (λ_n) *and* (μ_n) are increasing sequences such that $\lambda_n \ge \mu_n \ge 0$ (λ_n *and* μ_n are not both simultaneously equal to zero).

Let us choose

$$\phi(z)=\frac{2(z-w)-1}{(z-w)\left(1-(z-w)\right)^{2}}=-\frac{1}{z-w}+\sum_{n=1}^{\infty}n\left(z-w\right)^{n}$$

and

$$\varphi(z)=\frac{z^{2}-z(2w+1)+w(w+1)+1}{(z-w)\left(1-z+w\right)}=\frac{1}{z-w}+\sum_{n=1}^{\infty}\left(z-w\right)^{n},$$

in view of the convolution defined by (14), and performing simple calculations, we observe that

$$\left(f*\phi\right)(z)=(z-w)f'(z)$$

and

$$\left(f*\varphi\right)(z)=f(z).$$

Thus, the class $\sum_{w}^{*}\left(\phi,\varphi,\alpha,\beta,k\right)$ reduces to $\sum_{w}^{*}(\alpha)$ satisfying the relationship

$$\sum\nolimits_{w}^{*}\left(\frac{2(z-w)-1}{(z-w)\left(1-(z-w)\right)^{2}},\frac{z^{2}-z(2w+1)+w(w+1)+1}{(z-w)\left(1-z+w\right)},\alpha,1,0\right)=\sum\nolimits_{w}^{*}(\alpha).$$

Similarly, by putting

$$\phi(z)=\frac{4z^{2}-z\left(3+8w\right)+w\left(4w+3\right)+1}{(z-w)\left(1-(z-w)\right)^{3}}=\frac{1}{z-w}+\sum_{n=1}^{\infty}n^{2}\left(z-w\right)^{n}$$

and

$$\varphi(z)=\frac{2(z-w)-1}{(z-w)\left(1-(z-w)\right)^{2}}=-\frac{1}{z-w}+\sum_{n=1}^{\infty}n\left(z-w\right)^{n},$$

then in view of the convolution defined by (14), we find that

$$\left(f*\phi\right)(z)=(z-w)f'(z)+(z-w)^{2}f''(z)$$

and

$$\left(f*\varphi\right)(z)=(z-w)f'(z).$$

The class $\sum_{w}^{*}\left(\phi,\varphi,\alpha,\beta,k\right)$ reduces to $\sum_{C_w}(\alpha)$ and satisfies the relation

$$\sum\nolimits_{w}^{*}\left(\frac{4z^{2}-z\left(3+8w\right)+w\left(4w+3\right)+1}{(z-w)\left(1-(z-w)\right)^{3}},\frac{2(z-w)-1}{(z-w)\left(1-(z-w)\right)^{2}},\alpha,1,0\right)=\sum\nolimits_{C_w}(\alpha)$$

It is easy to check

$$\sum\nolimits_{0}^{*}\left(\frac{2z-1}{z\left(1-z\right)^{2}},\frac{z^{2}-z+1}{z\left(1-z\right)},\alpha,1,0\right)=\sum\nolimits^{*}(\alpha,1)$$

is the class of meromorfically starlike functions of order α,

$$\sum\nolimits_{0}^{*}\left(\frac{2z-1}{z\left(1-z\right)^{2}},\frac{z^{2}-z+1}{z\left(1-z\right)},0,1,0\right)=\sum\nolimits^{*}(0,1)$$

gives the whole class of meromorfically starlike functions whereas

$$\sum\nolimits_{0}^{*}\left(\frac{2z-1}{z\left(1-z\right)^{2}},\frac{z^{2}-z+1}{z\left(1-z\right)},0,\beta,0\right)=\sum\nolimits^{*}(0,\beta)$$

yields the class studied by Padmanabhan (1968).

Lastly, Darus (2004) defined the class

$$\sum\nolimits_{0}^{*}\left(\frac{2z-1}{z\left(1-z\right)^{2}},\frac{z^{2}-z+1}{z\left(1-z\right)},\alpha,\beta,k\right)=\sum\nolimits^{*}(\alpha,\beta,k).$$

For the class

$$\sum\nolimits_{0}^{*}\left(\frac{2z-1}{z\left(1-z\right)^{2}},\frac{z^{2}-z+1}{z\left(1-z\right)},\alpha,1,k\right)=\sum\nolimits^{*}(\alpha,k),$$

Owa and Pascu (2003) showed the following theorem.

Theorem 1

Let the function $f(z)$ be defined by

$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n}z^{n}$. If

$$\sum_{n=0}^{\infty}\left(n+k+\left|2\alpha+n-k\right|\right)\left|a_{n}\right|r^{n+1}\leq 2\left(1-\alpha\right)$$

for some $k(0\leq k\leq 1)$ and $\alpha(0\leq\alpha<1)$, then

$f(z)\in\sum^{*}(\alpha,k)$.

The result given by Darus (2004) for functions $f(z)\in\sum^{*}\left(\alpha,\beta,k\right)$ is given as the following theorem:

Theorem 2

Let the function $f(z)$ be defined by

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n$$. If

$$\sum_{n=0}^{\infty}\left(n+k+\beta\left|2\alpha+n-k\right|\right)\left|a_n\right|r^{n+1}\leq\beta\left(k+1-2\alpha\right)+1-k$$

for some $k(0\leq k\leq 1)$, $\alpha(0\leq\alpha<1)$ and $\beta(0<\beta\leq 1)$, then $f(z)\in\sum^{*}(\alpha,\beta,k)$.

COEFFICIENT INEQUALITIES

Our first result for functions $f(z)\in\sum_{w}^{*}(\phi,\varphi,\alpha,\beta,k)$ is given as the following theorem:

Theorem 3

Let the function $f(z)$ be defined by (11). If

$$\sum_{n=1}^{\infty}\left[\left(\lambda_n+k\mu_n\right)+\beta\left|\lambda_n+\left(2\alpha-k\right)\mu_n\right|\right]\left[1+(n-1)t\right]^m\left|a_n\right|\leq\left|\xi\right|\left(\beta\left|c+\left(2\alpha-k\right)d\right|-\left|c\right|+\left|d\right|k\right) \qquad (15)$$

where, $\lambda_n\geq\mu_n\geq 0$, $k(0\leq k\leq 1)$, $\alpha(0\leq\alpha<1)$, $\beta(0<\beta\leq 1)$, $c,d\in\mathbb{C}-\{0\}$, $t\geq 0$ and $\xi\in\mathbb{C}-\{0\}$ then $f(z)\in\sum_{w}^{*}(\phi,\varphi,\alpha,\beta,k)$.

Proof

Suppose (15) holds true, then it is sufficient to show that

$$\left|I_{t,\xi}^{m}(f*\phi)+kI_{t,\xi}^{m}(f*\varphi)\right|-\beta\left|I_{t,\xi}^{m}(f*\phi)+\left(2\alpha-k\right)I_{t,\xi}^{m}(f*\varphi)\right|$$
$$=\left|\xi(c+dk)+\sum_{n=1}^{\infty}\left[1+(n-1)t\right]^m\left(\lambda_n+k\mu_n\right)a_n(z-w)^n\right|-\beta\left|\xi\left[c+d\left(2\alpha-k\right)\right]+\sum_{n=1}^{\infty}\left[1+(n-1)t\right]^m\left[\lambda_n+\left(2\alpha-k\right)\mu_n\right]a_n(z-w)^n\right|.$$

$$\leq 0$$

So, it gives following relations

$$\leq\left|\xi\right|\left(\left|c\right|-\left|d\right|k\right)+\sum_{n=1}^{\infty}\left[1+(n-1)t\right]^m\left(\lambda_n+k\mu_n\right)\left|a_n\right|r^{n+1}-\beta\left|\xi\right|\left|\left[c+d\left(2\alpha-k\right)\right]\right|+\sum_{n=1}^{\infty}\beta\left[1+(n-1)t\right]^m\left|\left[\lambda_n+\left(2\alpha-k\right)\mu_n\right]\right|\left|a_n\right|r^{n+1}$$

$$=\left|\xi\right|\left(\left|c\right|-\left|d\right|k\right)-\beta\left|\xi\right|\left|\left[c+d\left(2\alpha-k\right)\right]\right|+\sum_{n=1}^{\infty}\left\{\left[1+(n-1)t\right]^m\left[\left(\lambda_n+k\mu_n\right)+\beta\left|\lambda_n+\left(2\alpha-k\right)\mu_n\right|\right]\right\}\left|a_n\right|r^{n+1}$$
$$\leq 0$$

These relations yield the following coefficient inequality:

$$\sum_{n=1}^{\infty}\left\{\left[1+(n-1)t\right]^m\left[\left(\lambda_n+k\mu_n\right)+\beta\left|\lambda_n+\left(2\alpha-k\right)\mu_n\right|\right]\right\}\left|a_n\right|\leq\left|\xi\right|\left(\beta\left|\left[c+d\left(2\alpha-k\right)\right]\right|-\left|c\right|+\left|d\right|k\right).$$

Hence $f(z)\in\sum_{w}^{*}(\phi,\varphi,\alpha,\beta,k)$. This completes the proof of the theorem.

Theorem 3 yields the following results:

Remark

The coefficient bound in the inequality of (15) is sharp for the function

$$f_n(z)=\frac{\xi}{z-w}+\frac{|\xi|\left(\beta\left|c+(2\alpha-k)d\right|-|c|+|d|k\right)}{\left[(\lambda_n+k\mu_n)+\beta\left|\lambda_n+(2\alpha-k)\mu_n\right|\right][1+(n-1)t]^m}(z-w)^n \quad (16)$$

Corollary

If the function $f\in\sum_w$ belongs to the class $\sum_w^*(\phi,\varphi,\alpha,\beta,k)$, then

$$|a_n|\le\frac{|\xi|\left(\beta\left|c+(2\alpha-k)d\right|-|c|+|d|k\right)}{\left[(\lambda_n+k\mu_n)+\beta\left|\lambda_n+(2\alpha-k)\mu_n\right|\right][1+(n-1)t]^m}, \quad n\ge 1 \quad (17)$$

where the equality holds true for the functions $f_n(z)$ given by (16).

Proof

Since $f(z)\in\sum_w^*(\phi,\varphi,\alpha,\beta,k)$, Theorem 3 gives

$$\sum_{n=1}^{\infty}\left[(\lambda_n+k\mu_n)+\beta\left|\lambda_n+(2\alpha-k)\mu_n\right|\right][1+(n-1)t]^m|a_n|\le|\xi|\left(\beta\left|c+(2\alpha-k)d\right|-|c|+|d|k\right)$$

Next, note that

$$\left[(\lambda_n+k\mu_n)+\beta\left|\lambda_n+(2\alpha-k)\mu_n\right|\right][1+(n-1)t]^m|a_n|\le\sum_{n=1}^{\infty}\left[(\lambda_n+k\mu_n)+\beta\left|\lambda_n+(2\alpha-k)\mu_n\right|\right][1+(n-1)t]^m|a_n|.$$

So,

$$|a_n|\le\frac{|\xi|\left(\beta\left|c+(2\alpha-k)d\right|-|c|+|d|k\right)}{\left[(\lambda_n+k\mu_n)+\beta\left|\lambda_n+(2\alpha-k)\mu_n\right|\right][1+(n-1)t]^m}$$

, $n\ge 1.$

DISTORTION PROPERTY

Now we give distortion property for the class $\Sigma_w^*(\phi,\varphi,\alpha,\beta,k)$ as follows:

Theorem 4

If the function $f(z)$ defined by (11) is in the class $\Sigma_w^*(\phi,\varphi,\alpha,\beta,k)$, then for $0<|z-w|=r<1,$

$$\frac{|\xi|}{r}-\frac{\left\{|\xi|\left(\beta\left|c+(2\alpha-k)d\right|-|c|+|d|k\right)\right\}r}{(\lambda_1+k\mu_1)+\beta\left|\lambda_1+(2\alpha-k)\mu_1\right|}\le|f(z)|\le\frac{|\xi|}{r}+\frac{\left\{|\xi|\left(\beta\left|c+(2\alpha-k)d\right|-|c|+|d|k\right)\right\}r}{(\lambda_1+k\mu_1)+\beta\left|\lambda_1+(2\alpha-k)\mu_1\right|},$$

and

$$\frac{|\xi|}{r^2}-\frac{|\xi|\left(\beta\left|c+(2\alpha-k)d\right|-|c|+|d|k\right)}{(\lambda_1+k\mu_1)+\beta\left|\lambda_1+(2\alpha-k)\mu_1\right|}\le|f'(z)|\le\frac{|\xi|}{r^2}+\frac{|\xi|\left(\beta\left|c+(2\alpha-k)d\right|-|c|+|d|k\right)}{(\lambda_1+k\mu_1)+\beta\left|\lambda_1+(2\alpha-k)\mu_1\right|}$$

are obtained.

Proof

Since $f\in\Sigma_w^*(\phi,\varphi,\alpha,\beta,k)$, Theorem 3 yields the inequality

$$\sum_{n=1}^{\infty}|a_n|\le\frac{|\xi|\left(\beta\left|c+(2\alpha-k)d\right|-|c|+|d|k\right)}{(\lambda_1+k\mu_1)+\beta\left|\lambda_1+(2\alpha-k)\mu_1\right|} \quad (18)$$

Thus, for $0<|z-w|=r<1$ and making use of (18), we have

$$|f(z)|\le\left|\frac{\xi}{z-w}\right|+\sum_{n=1}^{\infty}|a_n||(z-w)|^n\le\frac{|\xi|}{r}+r\sum_{n=1}^{\infty}|a_n|\le\frac{|\xi|}{r}+\frac{|\xi|\left(\beta|c+(2\alpha-k)d|-|c|+|d|k\right)}{(\lambda_1+k\mu_1)+\beta|\lambda_1+(2\alpha-k)\mu_1|}r,$$

and

$$|f(z)|\ge\left|\frac{\xi}{z-w}\right|-\sum_{n=1}^{\infty}|a_n||(z-w)|^n\ge\frac{|\xi|}{r}-r\sum_{n=1}^{\infty}|a_n|\ge\frac{|\xi|}{r}-\frac{|\xi|\left(\beta|c+(2\alpha-k)d|-|c|+|d|k\right)}{(\lambda_1+k\mu_1)+\beta|\lambda_1+(2\alpha-k)\mu_1|}r.$$

Also, from Theorem 3, we can obtain

$$\sum_{n=1}^{\infty}n|a_n|\le\frac{|\xi|\left(\beta|c+(2\alpha-k)d|-|c|+|d|k\right)}{(\lambda_1+k\mu_1)+\beta|\lambda_1+(2\alpha-k)\mu_1|}.$$

Hence

$$|f'(z)|\le\left|\frac{\xi}{(z-w)^2}\right|+\sum_{n=1}^{\infty}n|a_n||(z-w)|^{n-1}\le\frac{|\xi|}{r^2}+\sum_{n=1}^{\infty}n|a_n|\le\frac{|\xi|}{r^2}+\frac{|\xi|\left(\beta|c+(2\alpha-k)d|-|c|+|d|k\right)}{(\lambda_1+k\mu_1)+\beta|\lambda_1+(2\alpha-k)\mu_1|},$$

and

$$|f'(z)|\ge\left|\frac{\xi}{(z-w)^2}\right|-\sum_{n=1}^{\infty}n|a_n||(z-w)|^{n-1}\ge\frac{|\xi|}{r^2}-\sum_{n=1}^{\infty}n|a_n|\ge\frac{|\xi|}{r^2}-\frac{|\xi|\left(\beta|c+(2\alpha-k)d|-|c|+|d|k\right)}{(\lambda_1+k\mu_1)+\beta|\lambda_1+(2\alpha-k)\mu_1|}.$$

Thus, the proof of Theorem 4 is completed.

RADII OF STARLIKENESS AND CONVEXITY

We obtain the radius of starlikeness and convexity for the class $\Sigma_w^*(\phi,\varphi,\alpha,\beta,k)$.

Theorem 5

If the function $f(z)$ defined by (11) is in the class $\Sigma_w^*(\phi,\varphi,\alpha,\beta,k)$, then $f(z)$ is meromorphically starlike of order ρ $(0\le\rho<1)$ in the disc $|z-w|<r_1$, where r_1 is the largest value for which

$$r_1=\inf_{n\ge1}\left\{\frac{|\xi|(1-\rho)\left[(\lambda_n+k\mu_n)+\beta|\lambda_n+(2\alpha-k)\mu_n|\right]}{(n+2-\rho)\left[|\xi|\left(\beta|c+(2\alpha-k)d|-|c|+|d|k\right)\right]}\right\}^{1/(n+1)}.$$

The result is sharp for functions $f_n(z)$ given by (16).

Proof

It suffices to obtain that

$$\left|\frac{(z-w)\left(I_{t,\xi}^m f(z)\right)'}{I_{t,\xi}^m f(z)}+1\right|\le1-\rho.$$

for $|z-w|<r_1,$ we have

$$\left|\frac{(z-w)\left(I_{t,\xi}^m f(z)\right)'}{I_{t,\xi}^m f(z)}+1\right|\le\frac{\sum_{n=1}^{\infty}(n+1)[1+(n-1)t]^m|a_n||z-w|^{n+1}}{|\xi|-\sum_{n=1}^{\infty}[1+(n-1)t]^m|a_n||z-w|^{n+1}}\le1-\rho \tag{19}$$

Hence (19) holds true if

$$\sum_{n=1}^{\infty}(n+1)\left[1+(n-1)t\right]^m |a_n||z-w|^{n+1} \le (1-\rho)\left(|\xi| - \sum_{n=1}^{\infty}\left[1+(n-1)t\right]^m |a_n||z-w|^{n+1}\right)$$

or

$$\frac{\sum_{n=1}^{\infty}(n+2-\rho)\left[1+(n-1)t\right]^m |a_n||z-w|^{n+1}}{|\xi|(1-\rho)} \le 1 \qquad (20)$$

with the aid of (17), (20) is true for

$$\frac{(n+2-\rho)\left[1+(n-1)t\right]^m |z-w|^{n+1}}{|\xi|(1-\rho)} \le \frac{\left[1+(n-1)t\right]^m\left[(\lambda_n+k\mu_n)+\beta|\lambda_n+(2\alpha-k)\mu_n|\right]}{|\xi|\left(\beta|c+(2\alpha-k)d|-|c|+|d|k\right)}. \qquad (21)$$

Solving (21) for $|z-w|$, we obtain

$$|z-w| \le \left\{\frac{|\xi|(1-\rho)\left[(\lambda_n+k\mu_n)+\beta|\lambda_n+(2\alpha-k)\mu_n|\right]}{(n+2-\rho)\left[|\xi|\left(\beta|c+(2\alpha-k)d|-|c|+|d|k\right)\right]}\right\}^{1/(n+1)}.$$

This completes the proof of Theorem 5.

Theorem 6

If the function $f(z)$ defined by (11) is in the class $\Sigma_w^*(\phi,\varphi,\alpha,\beta,k)$, then $f(z)$ is meromorphically convex of order ρ $(0 \le \rho < 1)$ in the disc $|z-w| < r_2$, where r_2 is the largest value for which

$$r_2 = \inf_{n\ge 1}\left\{\frac{|\xi|(1-\rho)\left[(\lambda_n+k\mu_n)+\beta|\lambda_n+(2\alpha-k)\mu_n|\right]}{n(n+2-\rho)\left[|\xi|\left(\beta|c+(2\alpha-k)d|-|c|+|d|k\right)\right]}\right\}^{1/(n+1)}.$$

The result is sharp for functions $f_n(z)$ given by (16).

Proof

We omit the details of the proof. It sufficies to prove that

$$\left|\frac{(z-w)\left(I_{t,\xi}^m f(z)\right)''}{\left(I_{t,\xi}^m f(z)\right)'} + 2\right| \le 1-\rho$$

for $|z-w| \le r_2$, with the aid of Theorem 3. Hence we have the assertion of Theorem 6.

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