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The improved generalized Riccati equation mapping method and its application for solving a nonlinear partial differential equation (PDE) describing the dynamics of ionic currents along microtubules

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In this paper we apply the improved Riccati equation mapping method to construct many families of exact solutions of a nonlinear partial differential equation involving parameters of a special interest in nanobiosciences and biophysics which describe a model of microtubules as nonlinear RLC transmission lines. As results, we can successfully recover the previously known results that have been found using other methods. This method is straightforward and concise, and it can be applied to other nonlinear PDEs in mathematical physics. Comparison between our new results and the well-known results are given. Some comments on the well-known results are also presented at the end of this article.

Key words: Improved Riccati equation mapping method, exact traveling wave solutions, nonlinear partial differential equations (PDEs) of microtubules, Nonlinear RLC transmission lines.

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INTRODUCTION

In recent years, the exact traveling wave solutions for nonlinear partial differential equations (PDEs) has been investigated by many authors who are interested in non linear physical phenomena. Many powerful methods have been presented, such as the inverse scattering transform method (Ablowitz and Clarkson, 1991), the Hirota's bilinear method (Hirota, 1971), the Painleve expansion method (Weiss et al., 1983; Kudryashov, 1988, 1990, 1991), the Backlund truncated method (Miura, 1978; Rogers and Shadwick, 1982), the exp-function method (He and Wu, 2006; Yusufoglu, 2008; Zhang, 2008; Bekir, 2009, 2010; Aslan, 2011), the tanh-function method (Abdou, 2007; Fan, 2000; Zhang and Xia, 2008; Yusufoglu and Bekir, 2008), the Jacobi elliptic function

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method (Chen and Wang, 2005; Liu et al., 2001; Lu, 2005), the (G'/G)-expansion method (Wang et al., 2008; Zhang et al., 2008; Zayed, 2009, 2010; Bekir, 2008; Ayhan and Bekir, 2012, Kudryashov, 2010a,b; Aslan, 2010, 2011, 2012a,b), the generalized Riccati equation mapping method (Zhu 2008; Zayed and Arnous, 2013, Zayed et al., 2013), and so on.

In the present paper, we shall use the improved Riccati equation mapping method to find the exact solutions of a nonlinear PDE of nanobiosciences. The main idea of this method is that the traveling wave solutions of nonlinear equations can be expressed by polynomials in Q, where $Q = Q(\xi)$ satisfies the generalized Riccati equation $Q' = r + pQ + qQ^2$ where $\xi = kx + \omega t$, where r, p, k, ω and q are constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in the given nonlinear equation, the coefficients of this polynomial can obtained by solving a set of algebraic equations resulted from the process of using the proposed method.

The objective of this paper is to apply the improved Riccati equation mapping method for finding many families of exact traveling wave solutions of the following nonlinear PDE of special interest in nanobiosciences, namely, the transmission line models for microtubules as nonlinear RLC transmission line (Sekulic et al., 2011a, Sataric et al., 2010):

$$R_2 C_0 L^2 u_{xxt} + L^2 u_{xx} + 2R_1 C_0 \delta u u_t - R_1 C_0 u_t = 0 \quad (1)$$

where $R_1 = 10^9 \Omega$ and $R_2 = 7\times 10^6 \Omega$ stand for longitudinal and transversal component of resistance of an Elementary rings and parameter $\delta(\delta < 1)$ describes nonlinearity of ER capacitor in MT. Here $L = 8\times 10^{-9} m$ while $C_0 = 1.8\times 10^{-15} F$ is the total maximal capacitance of the ER. The physical details of the derivation of Equation (1) can be elaborated in Sataric et al. (2010). For further references about electrical models of microtubules, see for example Ilic et al. (2009), Sekulic et al. (2011b, 2012), Sataric et al. (2009); Freedman et al. (2010), and Sekulic and Sataric (2012). Recently, Equation (1) has been discussed in (Sekulic et al. 2011a) by using the modified extended tanh-function method, where its exact solutions have been found.

The rest of this paper can be organized as follows: First is description of the improved generalized Riccati equation method. Many families of exact traveling wave solutions for Equation (1) are next obtained. This is followed by illustrations on physical explanations for some obtained results. Thereafter, conclusions and comments on Sekulic et al. (2011a) as well as comparison between our new results and the well-known results obtained in Sekulic et al. (2011a) are investigated.

Description of the improved generalized Riccati equation mapping method

We suppose that a nonlinear PDE is in the following from:

$$P(u, u_x, u_t, u_{xx}, u_{tt}, ...) = o \quad (2)$$

where $u = u(x,t)$ is an unknown function , *P* is a polynomial in $u = u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. Let us now give the main steps for solving Equation (2.1) using the improved Riccati equation mapping method (Zhu 2008; Zayed and Arnous, 2013; Zayed et al, 2013):

Step 1: We look for its traveling wave solution in the form

$$u(x,t) = u(\xi), \quad \xi = kx + \omega t \quad (3)$$

where k, ω are constants. Substituting (3) into Equation (2) gives the nonlinear ODE for $u(\xi)$ as follows:

$$H(u, u', u'', ...) = 0 \quad (4)$$

where H is a polynomial in $u(\xi)$ and its total derivatives $u', u'', u''', ...$ such that $u' = \frac{du}{d\xi}, u'' = \frac{d^2u}{d\xi^2}, ...$

Step 2: We suppose that the solution of the ODE (4) can be expressed as follows:

$$u(\xi) = \sum_{i=-m}^{m} a_i Q^i(\xi), \quad (5)$$

where $a_i (i = 0, \pm 1, \pm 2, ..., \pm m)$ are constants to be determined later such as $a_m \neq 0$ or $a_{-m} \neq 0$ and $Q = Q(\xi)$ is the solution of generalized Riccati equation

$$Q' = r + pQ + qQ^2 \quad (6)$$

where r, p and q are constants, such that $q \neq 0$.

Step 3: We determine the positive integer *m* in (5) by balancing the nonlinear terms and the highest order derivatives of $u(\xi)$ in Equation (4).

Step 4: Substituting (5) and along with Equation (6) into Equation (4) and then equating all the coefficients of $Q^i (i = 0, \pm 1, \pm 2, ..., \pm m)$ to zero yield a system of algebraic equations which can be solved by using the Maple or Mathematica to find the values of the constants $a_i(-m,...,m)$ and k,ω.

Step 5: It is well-known (Zhu 2008; Zayed and Arnous, 2013; Zayed et al., 2013) that Equation (6) has many families of solutions as follows:

Type 1: When $\Delta = p^2 - 4qr > 0$ and $pq \neq 0$ or $qr \neq 0$ we have

$$\Phi_1(\xi) = -\frac{1}{2q}[p + \sqrt{\Delta}\tanh(\frac{\sqrt{\Delta}}{2}\xi)],$$

$$\Phi_2(\xi) = -\frac{1}{2q}[p + \sqrt{\Delta}\coth(\frac{\sqrt{\Delta}}{2}\xi)],$$

$$\Phi_3(\xi) = -\frac{1}{2q}[p + \sqrt{\Delta}(\tanh(\sqrt{\Delta}\xi) \pm i\,\mathrm{sech}(\sqrt{\Delta}\xi))], \quad i\sqrt{-1}$$

$$\Phi_4(\xi) = -\frac{1}{2q}[p + \sqrt{\Delta}(\coth(\sqrt{\Delta}\xi) \pm \mathrm{csch}(\sqrt{\Delta}\xi))],$$

$$\Phi_5(\xi) = -\frac{1}{4q}[2p + \sqrt{\Delta}(\tanh(\frac{\sqrt{\Delta}}{4}\xi) \pm \coth(\frac{\sqrt{\Delta}}{4}\xi))],$$

$$\Phi_6(\xi) = \frac{1}{2q}[-p + \frac{\sqrt{\Delta(A^2+B^2)} - A\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)}{A\sinh(\sqrt{\Delta}\xi) + B}],$$

$$\Phi_7(\xi) = \frac{1}{2q}[-p - \frac{\sqrt{\Delta(B^2-A^2)} + A\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)}{A\sinh(\sqrt{\Delta}\xi) + B}],$$

where *A* and *B* are two non-zero real constants satisfying B^2-$A^2>0$,

$$\Phi_8(\xi) = \frac{2r\cosh(\frac{\sqrt{\Delta}}{2}\xi)}{\sqrt{\Delta}\sinh(\frac{\sqrt{\Delta}}{2}\xi) - p\cos h(\frac{\sqrt{\Delta}}{2}\xi)},$$

$$\Phi_9(\xi) = \frac{-2r\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{p\sinh(\frac{\sqrt{\Delta}}{2}\xi) - \sqrt{\Delta}\cosh(\frac{\sqrt{\Delta}}{2}\xi)},$$

$$\Phi_{10}(\xi) = \frac{2r\cosh(\frac{\sqrt{\Delta}}{2}\xi)}{\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi) - p\cosh(\sqrt{\Delta}\xi) \pm i\sqrt{\Delta}}, \quad i = \sqrt{-1}$$

$$\Phi_{11}(\xi) = \frac{2r\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{-p\sinh(\sqrt{\Delta}\xi) + \sqrt{\Delta}\cosh(\sqrt{\Delta}\xi) \pm \sqrt{\Delta}},$$

$$\Phi_{12}(\xi) = \frac{4r\sinh(\frac{\sqrt{\Delta}}{4}\xi)\cosh(\frac{\sqrt{\Delta}}{4}\xi)}{-2p\sinh(\frac{\sqrt{\Delta}}{4}\xi)\cosh(\frac{\sqrt{\Delta}}{4}\xi) + 2\sqrt{\Delta}\cosh^2(\frac{\sqrt{\Delta}}{2}\xi) - \sqrt{\Delta}},$$

Type 2: When $\Delta = p^2 - 4qr < 0$ and $pq \neq 0$ or $qr \neq 0$ we have

$$\Phi_{13}(\xi) = \frac{1}{2q}[-p + \sqrt{-\Delta}\tan(\frac{\sqrt{-\Delta}}{2}\xi)],$$

$$\Phi_{14}(\xi) = -\frac{1}{2q}[p + \sqrt{-\Delta}\cot(\frac{\sqrt{-\Delta}}{2}\xi)],$$

$$\Phi_{15}(\xi) = \frac{1}{2q}[-p + \sqrt{-\Delta}(\tan(\sqrt{-\Delta}\xi) \pm \sec(\sqrt{-\Delta}\xi))],$$

$$\Phi_{16}(\xi) = -\frac{1}{2q}[p + \sqrt{-\Delta}(\cot(\sqrt{-\Delta}\xi) \pm \csc(\sqrt{-\Delta}\xi))],$$

$$\Phi_{17}(\xi) = \frac{1}{4q}[-2p + \sqrt{-\Delta}(\tan(\frac{\sqrt{-\Delta}}{4}\xi) - \cot(\frac{\sqrt{-\Delta}}{4}\xi))],$$

$$\Phi_{18}(\xi) = \frac{1}{2q}[-p + \frac{\pm\sqrt{-\Delta(A^2-B^2)} - A\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)}{A\sin(\sqrt{-\Delta}\xi) + B}],$$

$$\Phi_{19}(\xi) = \frac{1}{2q}[-p - \frac{\pm\sqrt{-\Delta(A^2-B^2)} - A\sqrt{-\Delta}\sin(\sqrt{-\Delta}\xi)}{A\sin(\sqrt{-\Delta}\xi) + B}],$$

where *A* and *B* are two non-zero real constants satisfying A^2-$B^2>0$,

$$\Phi_{20}(\xi) = -\frac{2r\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{\sqrt{-\Delta}\sin(\frac{\sqrt{-\Delta}}{2}\xi) + p\cos(\frac{\sqrt{-\Delta}}{2}\xi)},$$

$$\Phi_{21}(\xi) = \frac{2r\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{-p\sin(\frac{\sqrt{-\Delta}}{2}\xi) + \sqrt{-\Delta}\cos(\frac{\sqrt{-\Delta}}{2}\xi)},$$

$$\Phi_{22}(\xi) = -\frac{2r\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{\sqrt{-\Delta}\sin(\sqrt{-\Delta}\xi) + p\cos(\sqrt{-\Delta}\xi) \pm \sqrt{-\Delta}},$$

$$\Phi_{23}(\xi) = \frac{2r\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{-p\sin(\sqrt{-\Delta}\xi) + \sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi) \pm \sqrt{-\Delta}},$$

$$\Phi_{24}(\xi) = \frac{4r\sin(\frac{\sqrt{-\Delta}}{4}\xi)\cos(\frac{\sqrt{-\Delta}}{4}\xi)}{-2p\sin(\frac{\sqrt{-\Delta}}{4}\xi)\cos(\frac{\sqrt{-\Delta}}{4}\xi) + 2\sqrt{-\Delta}\cos^2(\frac{\sqrt{-\Delta}}{2}\xi) - \sqrt{-\Delta}},$$

Type 3: When $r = 0$ *and* $pq \neq 0$ we have

$$\Phi_{25}(\xi) = \frac{-pd}{q[d + \cosh(p\xi) - \sinh(p\xi)]},$$

$$\Phi_{26}(\xi) = -\frac{p[\cosh(p\xi) + \sinh(p\xi)]}{q[d + \cosh(p\xi) + \sinh(p\xi)]},$$

where d is an arbitrary constant.

Type 4: When $r = p = 0 \, and \; q \neq 0$ we have

$$\Phi_{27}(\xi) = \frac{-1}{q\xi + c_1},$$

where c_1 is an arbitrary constant.

Step 6: Substituting the well known solutions of Equation (6) listed above in Step 5 into (5) we have many families of exact solutions of Equation (2).

MANY FAMILIES OF EXACT TRAVELING WAVE SOLUTIONS FOR EQUATION (1)

Here we apply the proposed improved generalized Riccati equation mapping method to find many families of exact traveling wave solutions of Equation (1). To the end we use the wave transformation

$$u(x,t) = u(\xi), \quad \xi = \frac{1}{L}x - \frac{c}{\tau}t, \tag{7}$$

where $\tau = R_1 C_0 = 1.32 \times 10^{-6} s$, and c is the dimensionless velocity of the wave, to reduce Equation (1) into the following ODE:

$$u'' - \frac{\alpha}{c}u' + \frac{\beta}{2}u^2 - \gamma u = 0 \tag{8}$$

where $\alpha = \frac{\tau}{R_2 C_0}, \; \beta = \frac{2R_1\delta}{R_2}, \; \gamma = \frac{R_1}{R_2}$.

By balancing $u'' with \; u^2$, we have m=2. Hence the formal solution of Equation (8) takes the form:

$$u(\xi) = a_2 Q^2 + a_1 Q + a_0 + a_{-1}Q^{-1} + a_{-2}Q^{-2} \tag{9}$$

where $a_2, \; a_1, a_0, \; a_{-1}, a_{-2}$ are constants to be determined, such that $a_{-2} \neq 0$ or $a_2 \neq 0$.

Inserting (9) with the aid of Equation (6) into Equation (8) we get the following system of algebraic equations:

$$Q^4 : \quad 6a_2 q^2 + \frac{\beta}{2}a_2^2 = 0,$$

$$Q^{-4} : \quad 6a_{-2} r^2 + \frac{\beta}{2}a_{-2}^2 = 0,$$

$$Q^3 : \quad 10a_2 pq + 2a_1 q^2 - \frac{2q\alpha}{c}a_2 + \beta a_1 a_2 = 0,$$

$$Q^{-3} : \quad 10a_{-2} pr + 2a_{-1} r^2 + \frac{2r\alpha}{c}a_{-2} + \beta a_{-1} a_{-2} = 0,$$

$$Q^2 : \quad 8a_2 qr + 3a_1 pq + 4a_2 p^2 - \frac{\alpha}{c}(2a_2 p + a_1 q) + \frac{\beta}{2}(a_1^2 + 2a_0 a_2) - \gamma a_2 = 0,$$

$$Q^{-2} : \quad 8a_{-2} qr + 3a_{-1} pr + 4a_{-2} p^2 + \frac{\alpha}{c}(2a_{-2} p + a_{-1} r) + \frac{\beta}{2}(a_{-1}^2 + 2a_0 a_{-2}) - \gamma a_{-2} = 0,$$

$$Q : \quad 6a_2 pr + 2a_{-1} qr + a_1 p^2 - \frac{\alpha}{c}(2a_2 r + a_1 p) + \beta(a_{-1} a_2 + a_0 a_1) - \gamma a_1 = 0,$$

$$Q^{-1} : \quad 6a_{-2} pq + 2a_{-1} qr + a_{-1} p^2 + \frac{\alpha}{c}(2a_{-2} q + a_{-1} p) + \beta(a_1 a_{-2} + a_0 a_{-1}) - \gamma a_{-1} = 0,$$

$$Q^0 : \quad 2a_2 r^2 + a_1 pr + a_{-1} pq + 2a_{-2} q^2 - \frac{\alpha}{c}(a_1 r - a_{-1} q) + \frac{\beta}{2}(a_0^2 + 2a_{-1}a_1 + 2a_{-2}a_2) - \gamma a_o = 0.$$

By solving these algebraic equations with the aid of Maple or Mathematica we have the following cases:

Case 1

$$p = p, q = q, r = -\frac{1}{24q}(\gamma - 6p^2), a_0 = \frac{3}{2\beta}(-2p^2 + \frac{10}{3}\frac{p\gamma c}{\alpha} + \gamma), \; a_1 = \frac{12q(\frac{\alpha}{5c} - p)}{\beta},$$
$$a_2 = \frac{-12q^2}{\beta}, c = \frac{\alpha}{5}\sqrt{\frac{6}{\gamma}}, \; a_{-1} = a_{-2} = 0$$

Case 2

$$p = p, q = q, r = -\frac{1}{24q}(\gamma - 6p^2), a_0 = \frac{1}{2\beta}(-6p^2 - \frac{10p\gamma c}{\alpha} + 3\gamma), \; a_{-1} = \frac{(\frac{\alpha}{5c} + p)(\gamma - 6p^2)}{2q\beta},$$
$$a_{-2} = \frac{-(\gamma - 6p^2)^2}{48\beta q^2}, c = \frac{\alpha}{5}\sqrt{\frac{6}{\gamma}}, a_1 = a_2 = 0$$

Case 3

$$p = 0, q = q, r = -\frac{\gamma}{96q}, a_0 = \frac{5\gamma}{4\beta}, \; a_1 = \frac{12q\alpha}{5\beta c}, a_{-1} = \frac{\gamma\alpha}{40q\beta c}, a_2 = \frac{-12q^2}{\beta}$$
$$a_{-2} = \frac{-\gamma^2}{768\beta q^2}, c = \frac{\alpha}{5}\sqrt{\frac{6}{\gamma}}.$$

Exact traveling wave solutions of Equation (1) for Case 1

By using the case 1 and according to the values of solutions of type 1 in the proposed method, we obtain the following exact traveling wave solutions for Equation (1):

$$u_1(x,t) = \frac{3\gamma}{2\beta} + 2p[\frac{5\gamma c}{2\alpha} - \frac{3\alpha}{5\beta c}] - \frac{\alpha\sqrt{6\gamma}}{5\beta c}\tanh(\sqrt{\frac{\gamma}{24}}\xi) - \frac{\gamma}{2\beta}\tanh^2(\sqrt{\frac{\gamma}{24}}\xi),$$

$$u_2(x,t) = \frac{3\gamma}{2\beta} + 2p[\frac{5\gamma c}{2\alpha} - \frac{3\alpha}{5\beta c}] - \frac{\alpha\sqrt{6\gamma}}{5\beta c}\coth(\sqrt{\frac{\gamma}{24}}\xi) - \frac{\gamma}{2\beta}\coth^2(\sqrt{\frac{\gamma}{24}}\xi),$$

$$u_3(x,t)=\frac{3\gamma}{2\beta}+2p[\frac{5\gamma c}{2\alpha}-\frac{3\alpha}{5\beta c}]-\frac{\alpha\sqrt{6\gamma}}{5\beta c}\left[\tanh(\sqrt{\frac{\gamma}{6}}\xi)\pm i\ \mathrm{sech}(\sqrt{\frac{\gamma}{6}}\xi)\right]$$
$$-\frac{\gamma}{2\beta}\left[\tanh(\sqrt{\frac{\gamma}{6}}\xi)\pm i\ \mathrm{sech}(\sqrt{\frac{\gamma}{6}}\xi)\right]^2,$$

$$u_4(x,t)=\frac{3\gamma}{2\beta}+2p[\frac{5\gamma c}{2\alpha}-\frac{3\alpha}{5\beta c}]-\frac{\alpha\sqrt{6\gamma}}{5\beta c}\left[\coth(\sqrt{\frac{\gamma}{6}}\xi)\pm i\ \mathrm{csch}(\sqrt{\frac{\gamma}{6}}\xi)\right]$$
$$-\frac{\gamma}{2\beta}\left[\coth(\sqrt{\frac{\gamma}{6}}\xi)\pm i\ \mathrm{csch}(\sqrt{\frac{\gamma}{6}}\xi)\right]^2,$$

$$u_5(x,t)=\frac{3\gamma}{2\beta}+p[\frac{5\gamma c}{\alpha\beta}-\frac{6\alpha}{5\beta c}]-\frac{3\alpha}{5\beta c}\sqrt{\frac{\gamma}{6}}\left[\tanh(\sqrt{\frac{\gamma}{96}}\xi)\pm\coth(\sqrt{\frac{\gamma}{96}}\xi)\right]$$
$$-\frac{\gamma}{8\beta}\left[\tanh(\sqrt{\frac{\gamma}{96}}\xi)\pm\coth(\sqrt{\frac{\gamma}{96}}\xi)\right]^2,$$

$$u_6(x,t)=\frac{3\gamma}{2\beta}+\frac{p}{\beta}[\frac{5\gamma c}{\alpha}-\frac{6\alpha}{5c}]+\frac{\alpha\sqrt{6\gamma}}{5\beta c}\left[\frac{(\sqrt{A^2+B^2}-A\cosh(\sqrt{\frac{\gamma}{6}}\xi)}{(A\sinh(\sqrt{\frac{\gamma}{6}}\xi)+B)}\right]-\frac{\gamma}{2\beta}\left[\frac{(\sqrt{A^2+B^2}-A\cosh(\sqrt{\frac{\gamma}{6}}\xi)}{(A\sinh(\sqrt{\frac{\gamma}{6}}\xi)+B)}\right]^2,$$

$$u_7(x,t)=\frac{3\gamma}{2\beta}+\frac{p}{\beta}[\frac{5\gamma c}{\alpha}-\frac{6\alpha}{5c}]+\frac{\alpha\sqrt{6\gamma}}{5\beta c}\left[\frac{(\sqrt{B^2-A^2}+A\cosh(\sqrt{\frac{\gamma}{6}}\xi)}{(A\sinh(\sqrt{\frac{\gamma}{6}}\xi)+B)}\right]-\frac{\gamma}{2\beta}\left[\frac{(\sqrt{B^2-A^2}+A\cosh(\sqrt{\frac{\gamma}{6}}\xi)}{(A\sinh(\sqrt{\frac{\gamma}{6}}\xi)+B)}\right]^2,$$

where *A* and *B* are two non-zero real constants satisfying $B^2-A^2>0$,

$$u_8(x,t)=\frac{3}{2\beta}[-2p^2+\frac{10p\gamma c}{3\alpha}+\gamma]+\frac{24qr}{\beta}\left(\frac{(-p+\frac{\alpha}{5c})\cosh(\sqrt{\frac{\gamma}{24}}\xi)}{(\sqrt{\frac{\gamma}{6}}\sinh(\sqrt{\frac{\gamma}{24}}\xi)-\sqrt{\frac{\gamma}{6}}\cosh(\sqrt{\frac{\gamma}{24}}\xi))}\right)$$
$$-\frac{48q^2r^2\cosh^2(\sqrt{\frac{\gamma}{24}}\xi)}{\beta\left(\sqrt{\frac{\gamma}{6}}\sinh(\sqrt{\frac{\gamma}{24}}\xi)-\sqrt{\frac{\gamma}{6}}\cosh(\sqrt{\frac{\gamma}{24}}\xi)\right)^2},$$

$$u_9(x,t)=\frac{3}{2\beta}[-2p^2+\frac{10p\gamma c}{3\alpha}+\gamma]-\frac{24qr}{\beta}\left(\frac{(-p+\frac{\alpha}{5c})\sinh(\sqrt{\frac{\gamma}{24}}\xi)}{(p\sinh(\sqrt{\frac{\gamma}{24}}\xi)-\sqrt{\frac{\gamma}{6}}\cosh(\sqrt{\frac{\gamma}{24}}\xi))}\right)$$
$$-\frac{48q^2r^2\sinh^2(\sqrt{\frac{\gamma}{24}}\xi)}{\beta\left(p\sinh(\sqrt{\frac{\gamma}{24}}\xi)-\sqrt{\frac{\gamma}{6}}\cosh(\sqrt{\frac{\gamma}{24}}\xi)\right)^2},$$

$$u_{10}(x,t)=\frac{3}{2\beta}[-2p^2+\frac{10p\gamma c}{3\alpha}+\gamma]+\frac{24qr}{\beta}\left(\frac{(-p+\frac{\alpha}{5c})\cosh(\sqrt{\frac{\gamma}{24}}\xi)}{(\sqrt{\frac{\gamma}{6}}\sinh(\sqrt{\frac{\gamma}{6}}\xi)-p\cosh(\sqrt{\frac{\gamma}{6}}\xi)\pm i\sqrt{\frac{\gamma}{6}})}\right)$$
$$-\frac{48q^2r^2\cosh^2(\sqrt{\frac{\gamma}{24}}\xi)}{\beta\left(\sqrt{\frac{\gamma}{6}}\sinh(\sqrt{\frac{\gamma}{6}}\xi)-p\cosh(\sqrt{\frac{\gamma}{6}}\xi)\pm i\sqrt{\frac{\gamma}{6}}\right)^2},$$

$$u_{11}(x,t)=\frac{3}{2\beta}[-2p^2+\frac{10p\gamma c}{3\alpha}+\gamma]+\frac{24qr}{\beta}\left(\frac{(-p+\frac{\alpha}{5c})\sinh(\sqrt{\frac{\gamma}{24}}\xi)}{(-p\sinh(\sqrt{\frac{\gamma}{6}}\xi)-\sqrt{\frac{\gamma}{6}}\cosh(\sqrt{\frac{\gamma}{6}}\xi)\pm\sqrt{\frac{\gamma}{6}})}\right)$$
$$-\frac{48q^2r^2\sinh^2(\sqrt{\frac{\gamma}{24}}\xi)}{\beta\left(-p\sinh(\sqrt{\frac{\gamma}{6}}\xi)-\sqrt{\frac{\gamma}{6}}\cosh(\sqrt{\frac{\gamma}{6}}\xi)\pm\sqrt{\frac{\gamma}{6}}\right)^2},$$

$$u_{12}(x,t)=\frac{3}{2\beta}[-2p^2+\frac{10p\gamma c}{3\alpha}+\gamma]+\frac{24qr}{\beta}\left(\frac{(-p+\frac{\alpha}{5c})\sinh(\sqrt{\frac{\gamma}{24}}\xi)}{(-p\sinh(\sqrt{\frac{\gamma}{24}}\xi)+\sqrt{\frac{\gamma}{6}}\cosh(\sqrt{\frac{\gamma}{24}}\xi))}\right)$$
$$-\frac{48q^2r^2\sinh^2(\frac{\sqrt{\gamma}}{2\sqrt{6}}\xi)}{\beta\left(-p\sinh(\sqrt{\frac{\gamma}{24}}\xi)+\sqrt{\frac{\gamma}{6}}\cosh(\sqrt{\frac{\gamma}{24}}\xi)\right)^2},$$

where $\xi=\frac{1}{L}x-\frac{\alpha}{5}\sqrt{\frac{6}{\gamma}}\frac{t}{\tau}$.

Remark 1. If $\gamma=6p^2$, then $r=0$ and $\Delta=p^2$. Consequently we have

$$a_0=\frac{12P^2}{\beta},\ ,\ a_2=\frac{-12q^2}{\beta},c=\frac{\alpha}{5p},\ a_{-1}=a_1=a_{-2}=0\cdot$$

According to the values of the solutions of type 3 in Sec.2, we obtain the following exact traveling wave solutions for Equation (1):

$$u_{13}(x,t)=\frac{12p^2}{\beta}-\frac{12q^2}{\beta}\left(\frac{pd}{q[d+\cosh(p\xi)-sinh(p\xi)]}\right)^2,$$

$$u_{14}(x,t)=\frac{12p^2}{\beta}-\frac{12q^2}{\beta}\left(\frac{p[\cosh(p\xi)+sinh(p\xi)]}{q[d+\cosh(p\xi)-sinh(p\xi)]}\right)^2,$$

where $\xi=\frac{1}{L}x-\frac{\alpha}{5p}\frac{t}{\tau}$.

Exact traveling wave solutions of Equation (1) for case 2.

By using the case 2 and according to the values of solutions of type 1 in the proposed method, we obtain the following exact traveling wave solutions for Equation (1):

$$u_1(x,t)=\frac{1}{2\beta}[-6p^2-\frac{10p\gamma c}{\alpha}+3\gamma]-\frac{(\gamma-6p^2)(p+\frac{\alpha}{5c})}{\beta}\left[p+\sqrt{\frac{\gamma}{6}}\tanh(\sqrt{\frac{\gamma}{24}}\xi)\right]^{-1}$$
$$-\frac{(\gamma-6p^2)^2}{12\beta}\left[p+\sqrt{\frac{\gamma}{6}}\tanh(\sqrt{\frac{\gamma}{24}}\xi)\right]^{-2}$$

$$u_2(x,t)=\frac{1}{2\beta}[-6p^2-\frac{10p\gamma c}{\alpha}+3\gamma]-\frac{(\gamma-6p^2)(p+\frac{\alpha}{5c})}{\beta}\left[p+\sqrt{\frac{\gamma}{6}}\coth(\sqrt{\frac{\gamma}{24}}\xi)\right]^{-1}-\frac{(\gamma-6p^2)^2}{12\beta}\left[p+\sqrt{\frac{\gamma}{6}}\coth(\sqrt{\frac{\gamma}{24}}\xi)\right]^{-2}$$

$$u_3(x,t)=\frac{1}{2\beta}[-6p^2-\frac{10p\gamma c}{\alpha}+3\gamma]-\frac{(\gamma-6p^2)(p+\frac{\alpha}{5c})}{\beta}\left[p+\sqrt{\frac{\gamma}{6}}[\tanh(\sqrt{\frac{\gamma}{6}}\xi)\pm i\,\mathrm{sech}(\sqrt{\frac{\gamma}{6}}\xi)\right]^{-1}-\frac{(\gamma-6p^2)^2}{12\beta}\left[p+\sqrt{\frac{\gamma}{6}}[\tanh(\sqrt{\frac{\gamma}{6}}\xi)\pm i\,\mathrm{sech}(\sqrt{\frac{\gamma}{6}}\xi)\right]^{-2}$$

$$u_4(x,t)=\frac{1}{2\beta}[-6p^2-\frac{10p\gamma c}{\alpha}+3\gamma]-\frac{(\gamma-6p^2)(p+\frac{\alpha}{5c})}{\beta}\left[p+\sqrt{\frac{\gamma}{6}}[\coth(\sqrt{\frac{\gamma}{6}}\xi)\pm \mathrm{csch}(\sqrt{\frac{\gamma}{6}}\xi)\right]^{-1}-\frac{(\gamma-6p^2)^2}{12\beta}\left[p+\sqrt{\frac{\gamma}{6}}[\coth(\sqrt{\frac{\gamma}{6}}\xi)\pm \mathrm{csch}(\sqrt{\frac{\gamma}{6}}\xi)\right]^{-2}$$

$$u_5(x,t)=\frac{1}{2\beta}[-6p^2-\frac{10p\gamma c}{\alpha}+3\gamma]-\frac{2(\gamma-6p^2)(p+\frac{\alpha}{5c})}{\beta}\left[2p+\sqrt{\frac{\gamma}{96}}\xi[\tanh(\sqrt{\frac{\gamma}{96}}\xi)\pm\coth(\sqrt{\frac{\gamma}{96}}\xi)\right]^{-1}-\frac{(\gamma-6p^2)^2}{3\beta}\left[2p+\sqrt{\frac{\gamma}{6}}\xi[\tanh(\sqrt{\frac{\gamma}{96}}\xi)\pm\coth(\sqrt{\frac{\gamma}{96}}\xi)\right]^{-2}$$

$$u_6(x,t)=\frac{1}{2\beta}[-6p^2-\frac{10p\gamma c}{\alpha}+3\gamma]+\frac{(\gamma-6p^2)(p+\frac{\alpha}{5c})}{\beta}\left[-p+\frac{\sqrt{\frac{\gamma}{6}}[\sqrt{A^2+B^2}-A\cosh(\sqrt{\frac{\gamma}{6}}\xi)]}{A\sinh(\sqrt{\frac{\gamma}{6}}\xi)+B}\right]^{-1}-\frac{(\gamma-6p^2)^2}{12\beta}\left[-p+\frac{\sqrt{\frac{\gamma}{6}}[\sqrt{A^2+B^2}-A\cosh(\sqrt{\frac{\gamma}{6}}\xi)]}{A\sinh(\sqrt{\frac{\gamma}{6}}\xi)+B}\right]^{-2}$$

$$u_7(x,t)=\frac{1}{2\beta}[-6p^2-\frac{10p\gamma c}{\alpha}+3\gamma]+\frac{(\gamma-6p^2)(p+\frac{\alpha}{5c})}{\beta}\left[-p-\frac{\sqrt{\frac{\gamma}{6}}[\sqrt{B^2-A^2}-A\cosh(\sqrt{\frac{\gamma}{6}}\xi)]}{A\sinh(\sqrt{\frac{\gamma}{6}}\xi)+B}\right]^{-1}-\frac{(\gamma-6p^2)^2}{12\beta}\left[-p-\frac{\sqrt{\frac{\gamma}{6}}[\sqrt{B^2-A^2}-A\cosh(\sqrt{\frac{\gamma}{6}}\xi)]}{A\sinh(\sqrt{\frac{\gamma}{6}}\xi)+B}\right]^{-2}$$

where A and B are two non-zero real constants satisfying $B^2-A^2>0$,

$$u_8(x,t)=\frac{1}{2\beta}[-6p^2-\frac{10p\gamma c}{\alpha}+3\gamma]-\frac{(\gamma-6p^2)(p+\frac{\alpha}{5c})}{4qr\beta}\left[\frac{\cosh(\sqrt{\frac{\gamma}{24}}\xi)}{\sqrt{\frac{\gamma}{6}}\sinh(\sqrt{\frac{\gamma}{24}}\xi)-p\cosh(\sqrt{\frac{\gamma}{24}}\xi)}\right]^{-1}-\frac{(\gamma-6p^2)^2}{192\beta q^2r^2}\left[\frac{\cosh(\sqrt{\frac{\gamma}{24}}\xi)}{\sqrt{\frac{\gamma}{6}}\sinh(\sqrt{\frac{\gamma}{24}}\xi)-p\cosh(\sqrt{\frac{\gamma}{24}}\xi)}\right]^{-2}$$

$$u_9(x,t)=\frac{1}{2\beta}[-6p^2-\frac{10p\gamma c}{\alpha}+3\gamma]-\frac{(\gamma-6p^2)(p+\frac{\alpha}{5c})}{4qr\beta}\left[\frac{\sinh(\sqrt{\frac{\gamma}{24}}\xi)}{p\sinh(\sqrt{\frac{\gamma}{24}}\xi)-\sqrt{\frac{\gamma}{6}}\cosh(\sqrt{\frac{\gamma}{24}}\xi)}\right]^{-1}-\frac{(\gamma-6p^2)^2}{192\beta q^2r^2}\left[\frac{\sinh(\sqrt{\frac{\gamma}{24}}\xi)}{p\sinh(\sqrt{\frac{\gamma}{24}}\xi)-\sqrt{\frac{\gamma}{6}}\cosh(\sqrt{\frac{\gamma}{24}}\xi)}\right]^{-2}$$

$$u_{10}(x,t)=\frac{1}{2\beta}[-6p^2-\frac{10p\gamma c}{\alpha}+3\gamma]+\frac{(\gamma-6p^2)(p+\frac{\alpha}{5c})}{4qr\beta}\left[\frac{\cosh(\sqrt{\frac{\gamma}{24}}\xi)}{\sqrt{\frac{\gamma}{6}}\sinh(\sqrt{\frac{\gamma}{6}}\xi)-p\cosh(\sqrt{\frac{\gamma}{6}}\xi)\pm i\sqrt{\frac{\gamma}{6}}}\right]^{-1}-\frac{(\gamma-6p^2)^2}{192\beta q^2r^2}\left[\frac{\cosh(\sqrt{\frac{\gamma}{24}}\xi)}{\sqrt{\frac{\gamma}{6}}\sinh(\sqrt{\frac{\gamma}{6}}\xi)-p\cosh(\sqrt{\frac{\gamma}{6}}\xi)\pm i\sqrt{\frac{\gamma}{6}}}\right]^{-2}$$

$$u_{11}(x,t)=\frac{1}{2\beta}[-6p^2-\frac{10p\gamma c}{\alpha}+3\gamma]+\frac{(\gamma-6p^2)(p+\frac{\alpha}{5c})}{4qr\beta}\left[\frac{\sinh(\sqrt{\frac{\gamma}{24}}\xi)}{-p\sinh(\sqrt{\frac{\gamma}{6}}\xi)+\sqrt{\frac{\gamma}{6}}\cosh(\sqrt{\frac{\gamma}{6}}\xi)\pm\sqrt{\frac{\gamma}{6}}}\right]^{-1}-\frac{(\gamma-6p^2)^2}{192\beta q^2r^2}\left[\frac{\sinh(\sqrt{\frac{\gamma}{24}}\xi)}{-p\sinh(\sqrt{\frac{\gamma}{6}}\xi)+\sqrt{\frac{\gamma}{6}}\cosh(\sqrt{\frac{\gamma}{6}}\xi)\pm\sqrt{\frac{\gamma}{6}}}\right]^{-2}$$

$$u_{12}(x,t)=\frac{1}{2\beta}[-6p^2-\frac{10p\gamma c}{\alpha}+3\gamma]+\frac{(\gamma-6p^2)(p+\frac{\alpha}{5c})}{4qr\beta}\left[\frac{\sinh(\sqrt{\frac{\gamma}{24}}\xi)}{-p\sinh(\sqrt{\frac{\gamma}{24}}\xi)+\sqrt{\frac{\gamma}{6}}\cosh(\sqrt{\frac{\gamma}{24}}\xi)}\right]^{-1}-\frac{(\gamma-6p^2)^2}{192\beta q^2r^2}\left[\frac{\sinh(\sqrt{\frac{\gamma}{24}}\xi)}{-p\sinh(\sqrt{\frac{\gamma}{24}}\xi)+\sqrt{\frac{\gamma}{6}}\cosh(\sqrt{\frac{\gamma}{24}}\xi)}\right]^{-2}$$

where $\xi=\frac{1}{L}x-\frac{\alpha}{5}\sqrt{\frac{\gamma}{6}}\frac{t}{\tau}$.

Exact traveling wave solutions of Equation (1) for Case 3

By using the case 3 and according to the values of solutions of type 1 in the proposed method, we obtain the following exact traveling wave solutions for Equation (1):

$$u_1(x,t)=\frac{5\gamma}{4\beta}-\frac{\alpha\sqrt{6\gamma}}{5\beta c}\tanh(\sqrt{\frac{\gamma}{24}}\xi)+\frac{12qr}{\beta}\tanh^2(\sqrt{\frac{\gamma}{24}}\xi)+\frac{\alpha\sqrt{6\gamma}}{20\beta c}\coth(\sqrt{\frac{\gamma}{24}}\xi)-\frac{\gamma}{32\beta}\coth^2(\sqrt{\frac{\gamma}{24}}\xi)$$

$$u_2(x,t)=\frac{5\gamma}{4\beta}-\frac{\alpha\sqrt{6\gamma}}{5\beta c}\coth(\sqrt{\frac{\gamma}{24}}\xi)+\frac{12qr}{\beta}\coth^2(\sqrt{\frac{\gamma}{24}}\xi)+\frac{\alpha\sqrt{6\gamma}}{20\beta c}\tanh(\sqrt{\frac{\gamma}{24}}\xi)-\frac{\gamma}{32\beta}\tanh^2(\sqrt{\frac{\gamma}{24}}\xi)$$

$$u_3(x,t)=\frac{5\gamma}{4\beta}-\frac{\alpha\sqrt{6\gamma}}{5\beta c}\left(\tanh(\sqrt{\frac{\gamma}{6}}\xi)\pm i\,\mathrm{sech}(\sqrt{\frac{\gamma}{6}}\xi)\right)+\frac{12qr}{\beta}\left(\tanh(\sqrt{\frac{\gamma}{6}}\xi)\pm i\,\mathrm{sech}(\sqrt{\frac{\gamma}{6}}\xi)\right)^2+\frac{\alpha\sqrt{6\gamma}}{20\beta c}\left(\tanh(\sqrt{\frac{\gamma}{6}}\xi)\pm i\,\mathrm{sech}(\sqrt{\frac{\gamma}{6}}\xi)\right)^{-1}-\frac{\gamma}{32\beta}\left(\tanh(\sqrt{\frac{\gamma}{6}}\xi)\pm i\,\mathrm{sech}(\sqrt{\frac{\gamma}{6}}\xi)\right)^{-2}$$

$$u_4(x,t)=\frac{5\gamma}{4\beta}-\frac{\alpha\sqrt{6\gamma}}{5\beta c}\left(\coth(\sqrt{\frac{\gamma}{6}}\xi)\pm\mathrm{csch}(\sqrt{\frac{\gamma}{6}}\xi)\right)+\frac{12qr}{\beta}\left(\coth(\sqrt{\frac{\gamma}{6}}\xi)\pm\mathrm{csch}(\sqrt{\frac{\gamma}{6}}\xi)\right)^2+\frac{\alpha\sqrt{6\gamma}}{20\beta c}\left(\coth(\sqrt{\frac{\gamma}{6}}\xi)\pm\mathrm{csch}(\sqrt{\frac{\gamma}{6}}\xi)\right)^{-1}+\frac{\gamma}{32\beta}\left(\coth(\sqrt{\frac{\gamma}{6}}\xi)\pm\mathrm{csch}(\sqrt{\frac{\gamma}{6}}\xi)\right)^{-2}$$

$$u_5(x,t)=\frac{5\gamma}{4\beta}-\frac{\alpha}{5\beta c}\sqrt{\frac{3\gamma}{2}}\left(\tanh(\sqrt{\frac{\gamma}{96}}\xi)\pm\coth(\sqrt{\frac{\gamma}{96}}\xi)\right)+\frac{3qr}{\beta}\left(\tanh(\sqrt{\frac{\gamma}{96}}\xi)\pm\coth(\sqrt{\frac{\gamma}{96}}\xi)\right)^2-\frac{\alpha\sqrt{6\gamma}}{10\beta c}\left(\tanh(\sqrt{\frac{\gamma}{96}}\xi)\pm\coth(\sqrt{\frac{\gamma}{96}}\xi)\right)^{-1}-\frac{\gamma}{8\beta}\left(\tanh(\sqrt{\frac{\gamma}{96}}\xi)\pm\coth(\sqrt{\frac{\gamma}{96}}\xi)\right)^{-2}$$

$$u_6(x,t)=\frac{5\gamma}{4\beta}+\frac{\alpha\sqrt{6\gamma}}{5\beta c}\left(\frac{\sqrt{A^2+B^2}-A\cosh(\sqrt{\frac{\gamma}{6}}\xi)}{A\sinh(2\sqrt{-qr}\,\xi)+B}\right)+\frac{12qr}{\beta}\left(\frac{\sqrt{A^2+B^2}-A\cosh(\sqrt{\frac{\gamma}{6}}\xi)}{A\sinh(\sqrt{\frac{\gamma}{6}}\xi)+B}\right)^2$$
$$+\frac{\alpha\sqrt{6\gamma}}{20\beta c}\left(\frac{\sqrt{A^2+B^2}-A\cosh(\sqrt{\frac{\gamma}{6}}\xi)}{A\sinh(\sqrt{\frac{\gamma}{6}}\xi)+B}\right)^{-1}-\frac{\gamma}{32\beta}\left(\frac{\sqrt{A^2+B^2}-A\cosh(\sqrt{\frac{\gamma}{6}}\xi)}{A\sinh(\sqrt{\frac{\gamma}{6}}\xi)+B}\right)^{-2}$$

$$u_7(x,t)=\frac{5\gamma}{4\beta}-\frac{\alpha\sqrt{6\gamma}}{5\beta c}\left(\frac{\sqrt{B^2-A^2}+A\cosh(\sqrt{\frac{\gamma}{6}}\xi)}{A\sinh(\sqrt{\frac{\gamma}{6}}\xi)+B}\right)+\frac{12qr}{\beta}\left(\frac{\sqrt{B^2-A^2}+A\cosh(\sqrt{\frac{\gamma}{6}}\xi)}{A\sinh(\sqrt{\frac{\gamma}{6}}\xi)+B}\right)^2$$
$$-\frac{\alpha\sqrt{6\gamma}}{20\beta c}\left(\frac{\sqrt{B^2-A^2}+A\cosh(\sqrt{\frac{\gamma}{6}}\xi)}{A\sinh(\sqrt{\frac{\gamma}{6}}\xi)+B}\right)^{-1}-\frac{\gamma}{32\beta}\left(\frac{\sqrt{B^2-A^2}+A\cosh(\sqrt{\frac{\gamma}{6}}\xi)}{A\sinh(\sqrt{\frac{\gamma}{6}}\xi)+B}\right)^{-2}$$

where *A* and *B* are two non-zero real constants satisfying $B^2-A^2>0$,

$$u_8(x,t)=\frac{5\gamma}{4\beta}-\frac{\alpha\sqrt{6\gamma}}{5\beta c}\coth(\sqrt{\frac{\gamma}{24}}\xi)+\frac{3q}{\beta r}\coth^2(\sqrt{\frac{\gamma}{24}}\xi)+\frac{\alpha\gamma}{20q\beta c}\sqrt{\frac{\gamma}{24}}\tanh(\sqrt{\frac{\gamma}{24}}\xi)+\frac{\gamma^2 r}{192\beta q}\tanh^2(\sqrt{\frac{\gamma}{24}}\xi)$$

$$u_9(x,t)=\frac{5\gamma}{4\beta}-\frac{\alpha\sqrt{6\gamma}}{5\beta c}\tanh(\sqrt{\frac{\gamma}{24}}\xi)+\frac{3q}{\beta r}\tanh^2(\sqrt{\frac{\gamma}{24}}\xi)+\frac{\alpha\gamma}{20q\beta c}\sqrt{\frac{\gamma}{24}}\coth(\sqrt{\frac{\gamma}{24}}\xi)+\frac{\gamma^2 r}{192\beta q}\coth^2(\sqrt{\frac{\gamma}{24}}\xi)$$

$$u_{10}(x,t)=\frac{5\gamma}{4\beta}-\frac{\alpha\sqrt{6\gamma}}{5\beta c}\left(\frac{\cosh(\sqrt{\frac{\gamma}{24}}\xi)}{\sinh(\sqrt{\frac{\gamma}{6}}\xi)\pm i}\right)+\frac{12qr}{\beta}\left(\frac{\cosh(\sqrt{\frac{\gamma}{24}}\xi)}{\sinh(\sqrt{\frac{\gamma}{6}}\xi)\pm i}\right)^2-\frac{\alpha\sqrt{6\gamma}}{20\beta c}\left(\frac{\coth(\sqrt{\frac{\gamma}{24}}\xi)}{\sinh(\sqrt{\frac{\gamma}{6}}\xi)\pm i}\right)^{-1}$$
$$-\frac{\gamma}{32\beta}\left(\frac{\cosh(\sqrt{\frac{\gamma}{24}}\xi)}{\sinh(\sqrt{\frac{\gamma}{6}}\xi)\pm i}\right)^{-2}$$

$$u_{11}(x,t)=\frac{5\gamma}{4\beta}-\frac{\alpha\sqrt{6\gamma}}{5\beta c}\left(\frac{\sinh(\sqrt{\frac{\gamma}{24}}\xi)}{coh(\sqrt{\frac{\gamma}{6}}\xi)\pm i}\right)+\frac{12qr}{\beta}\left(\frac{\sinh(\sqrt{\frac{\gamma}{24}}\xi)}{\cosh(\sqrt{\frac{\gamma}{6}}\xi)\pm i}\right)^2-\frac{\alpha\sqrt{6\gamma}}{20\beta c}\left(\frac{\sinh(\sqrt{\frac{\gamma}{24}}\xi)}{\cosh(\sqrt{\frac{\gamma}{6}}\xi)\pm i}\right)^{-1}$$
$$-\frac{\gamma}{32\beta}\left(\frac{\sinh(\sqrt{\frac{\gamma}{24}}\xi)}{\cosh(\sqrt{\frac{\gamma}{6}}\xi)\pm i}\right)^{-2}$$

$$u_{12}(x,t)=\frac{5\gamma}{4\beta}-\frac{\alpha\sqrt{6\gamma}}{5\beta c}\left(\frac{\sinh(\sqrt{\frac{\gamma}{24}}\xi)}{\cosh(\sqrt{\frac{\gamma}{24}}\xi)-1}\right)+\frac{12qr}{\beta}\left(\frac{\sinh(\sqrt{\frac{\gamma}{24}}\xi)}{\cosh(\sqrt{\frac{\gamma}{24}}\xi)-1}\right)^2-\frac{\alpha\sqrt{6\gamma}}{20\beta c}\left(\frac{\sinh(\sqrt{\frac{\gamma}{24}}\xi)}{\cosh(\sqrt{\frac{\gamma}{24}}\xi)-1}\right)^{-1}$$
$$-\frac{\gamma}{31\beta}\left(\frac{\sinh(\sqrt{\frac{\gamma}{24}}\xi)}{\cosh(\sqrt{\frac{\gamma}{24}}\xi)-1}\right)^{-2}$$

where, $\xi=\frac{1}{L}x-\frac{\alpha}{5}\sqrt{\frac{\gamma}{6}}\frac{t}{\tau}$.

Remark 2

We have noted in the two cases 1, 2 that $\Delta=\frac{\gamma}{6}>0$ while in case 3 we have $\Delta=\frac{\gamma}{24}>0$. This yields that all the solutions of type 2 in the proposed method cannot be considered in this paper. Therefore, Equation (1) has no trigonometric function solutions.

PHYSICAL EXPLANATIONS FOR SOME OBTAINED RESULTS

Here, we will present some graphs for the obtained solutions of Equation (1) by selecting some special values of the parameters in the exact solutions using the mathematical software Maple, which can be shown below in Figures 1 to 6. From these explicit solutions, we see that $u_1(x,t)$ in both cases 1, 2 are kink shaped soliton solutions while $u_2(x,t)$ in these two cases are singular kink shaped soliton solutions. The two solutions $u_1(x,t)$ and $u_2(x,t)$ in case 3 are kink-singular shaped soliton solutions. The graphical representations of these solutions are shown in the following figures:

CONCLUSIONS AND COMMENTS

Here we give some comments on the solutions (22) to (27) of the second model of microtubules (16) in Sekulic et al. (2011a). We have found that some of these solutions are incorrect. Thus, we will correct these solutions and then compare between some of our results in the present article and the corrected solutions as follows:

1) There is a minor error in Equation (18) of Sekulic et al. (2011a). The correction is to replace β by $\frac{\beta}{2}$ in Equation (18).

2) The results of Case I in Sekulic et al. (2011a) do not satisfy the algebraic Equation (21). After a careful revision, we have shown that the correction of this case should be in the form:

$$b=-\frac{\gamma}{24},\ a_0=\frac{3\gamma}{4\beta},\ \ a_1=\frac{6\alpha}{5\beta c},a_2=\frac{-6}{\beta},c=\frac{\alpha}{5}\sqrt{\frac{6}{\gamma}},\ b_1=b_2=0 \quad (10)$$

It is easy to see that the corrected values (10) satisfy the algebraic Equation (21) of Sekulic et al. (2011a).

Hence with replacing β by $\frac{\beta}{2}$ in (10), the result (22) of Sekulic et al. (2011a) should be rewritten in the corrected form:

$$u(x,t)=\frac{3\gamma}{2\beta}-\frac{12\alpha}{5\beta c}\sqrt{\frac{\gamma}{24}}\tanh\left(\sqrt{\frac{\gamma}{24}}\left(\frac{1}{L}x-\frac{\alpha}{5}\sqrt{\frac{\gamma}{6}}\frac{t}{\tau}\right)\right)-\frac{\gamma}{2\beta}\tanh^2\left(\sqrt{\frac{\gamma}{24}}\left(\frac{1}{L}x-\frac{\alpha}{5}\sqrt{\frac{\gamma}{6}}\frac{t}{\tau}\right)\right) \quad (11)$$

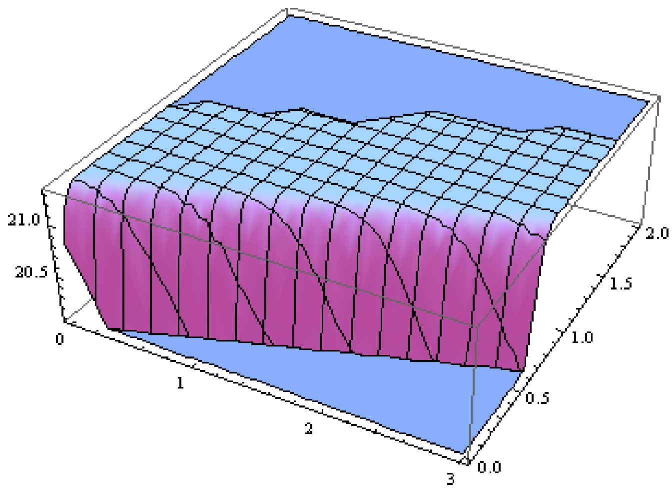

Figure 1. The plot of $u_1(x,t)$ of case 1, where $p=1, q=1, \alpha=104.76, \beta=142.86, \gamma=142.86.$

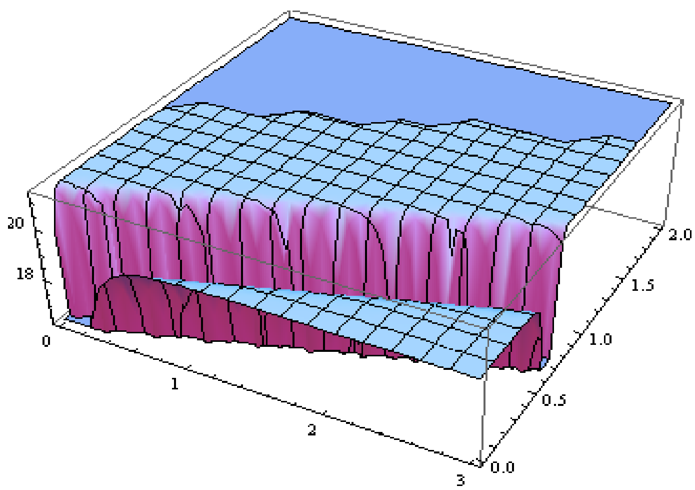

Figure 2. The plot of $u_2(x,t)$ of case 1, where $p=1, q=1, \alpha=104.76, \beta=142.86, \gamma=142.86.$

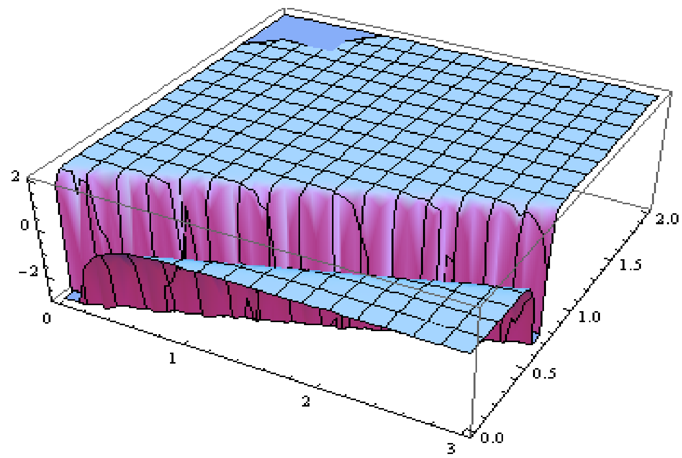

Figure 3. The plot of $u_1(x,t)$ of case 2, where $p=1, q=1, \alpha=104.76, \beta=142.86, \gamma=142.86.$

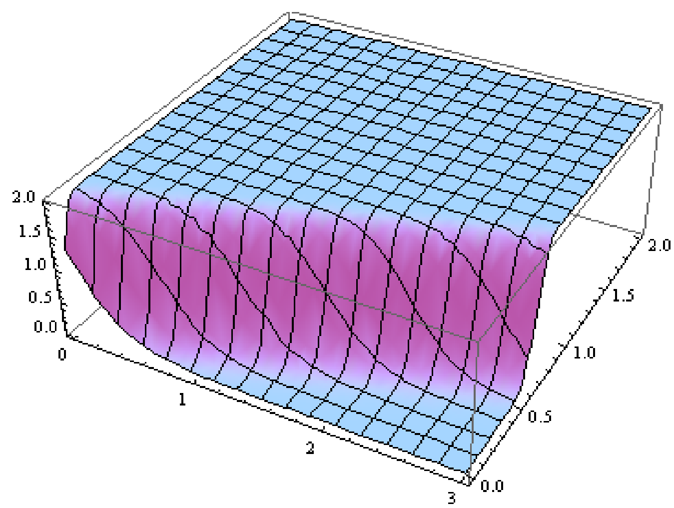

Figure 4. The plot of $u_2(x,t)$ of case 2, where $p=1, q=1, \alpha=104.76, \beta=142.86, \gamma=142.86.$

On comparing the corrected result (11) with our result $u_1(x,t)$ of "Exact traveling wave solutions of Equation (1) for case 1", we conclude that they are equivalent if $p=0$ and $q=1$.

3) The results of Case II in Sekulic et al. (2011a) do not satisfy the algebraic Equation (21) too. After a careful revision, we have shown that the correction of this case should be in the form:

$$b=\frac{\gamma}{24},\ a_0=\frac{\gamma}{4\beta},\ \ a_1=\frac{6\alpha}{5\beta c}, a_2=\frac{-6}{\beta}, c=\frac{\alpha}{5}\sqrt{\frac{-6}{\gamma}},\ b_1=b_2=0 \quad (12)$$

It is easy to see that the values of (12) satisfy the algebraic Equation (21) of Sekulic et al. (2011a). From the values of (12) we deduce that $\gamma=\frac{-6\alpha^2}{25c^2}$ which is negative. This contradicts that $\gamma=\frac{R_1}{R_2}>0$. Therefore, Case 2 in Sekulic et al. (2011a) should be rejected.

4) The results of Case III in Sekulic et al. (2011a) do not satisfy the algebraic Equation (21) too. After a careful revision, we have shown that the correction of this case should be in the form:

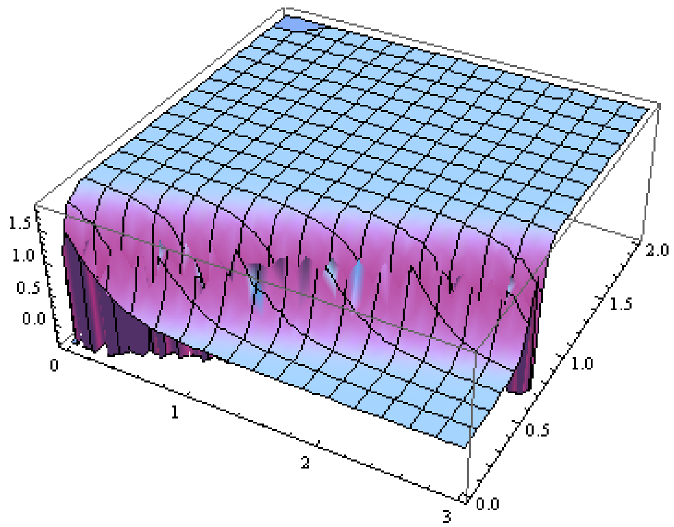

Figure 5. The plot of $u_1(x,t)$ of case 3, where $p=0, q=1, \alpha=104.76, \beta=142.86, \gamma=142.86$.

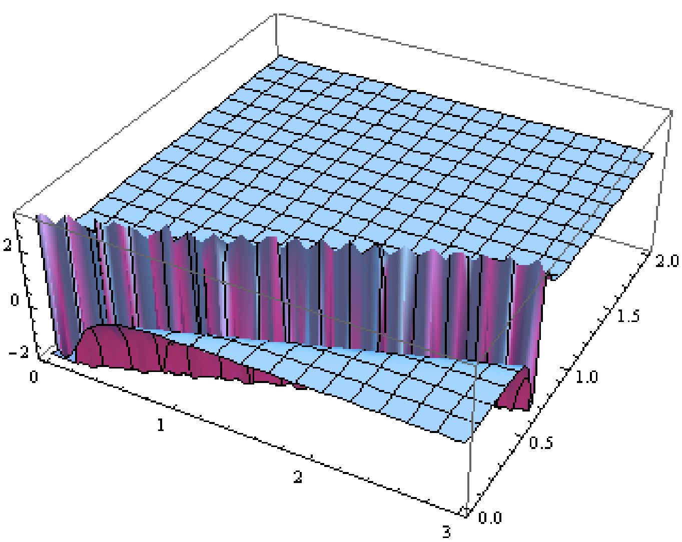

Figure 6. The plot of $u_2(x,t)$ of case 3, where $p=0, q=1, \alpha=104.76, \beta=142.86, \gamma=142.86$.

$$b=-\frac{\gamma}{24},\ a_0=\frac{3\gamma}{4\beta},\ b_1=\frac{\alpha\gamma}{20\beta c}, b_2=\frac{-\gamma^2}{96\beta}, c=\frac{\alpha}{5}\sqrt{\frac{6}{\gamma}},\ a_1=a_2=0 \qquad (13)$$

It is easy to see that the corrected values of (13) satisfy the algebraic Equation (21) of Sekulic et al. (2011a).

Hence with replacing β by $\frac{\beta}{2}$ in (13), the result (24) of (Sekulic et al, 2011a) should be rewritten in the corrected form:

$$u(x,t)=\frac{3\gamma}{2\beta}-\frac{\alpha\gamma}{10\beta c}\sqrt{\frac{24}{\gamma}}\coth\left(\sqrt{\frac{\gamma}{24}}\left(\frac{1}{L}x-\frac{\alpha}{5}\sqrt{\frac{6}{\gamma}}\frac{t}{\tau}\right)\right)-\frac{\gamma}{2\beta}\coth^2\left(\sqrt{\frac{\gamma}{24}}\left(\frac{1}{L}x-\frac{\alpha}{5}\sqrt{\frac{6}{\gamma}}\frac{t}{\tau}\right)\right) \qquad (14)$$

On comparing the corrected result (14) with our result $u_1(x,t)$ of 'Exact traveling wave solutions of Equation (1) for case 2', we conclude that they are equivalent if $p=0$ and $q=1$.

5) The results of Case IV of Sekulic et al. (2011a) still do not satisfy the algebraic Equation (21). After a careful revision, we have shown that the correction of this case should be in the form:

$$b=\frac{\gamma}{24},\ c=\frac{\alpha}{5}\sqrt{\frac{-6}{\gamma}},\ a_1=a_2=0,\ b_2=\frac{-\gamma^2}{96\beta},\ b_1=-\frac{\alpha\gamma}{20\beta c}, a_0=\frac{\gamma}{4\beta} \qquad (15)$$

It is easy to see that the corrected values of (15) satisfy the algebraic Equation (21) of Sekulic et al. (2011a). From the values of Equation (15) we deduce that $\gamma=\frac{-6\alpha^2}{25c^2}$ which is negative. This contradicts that $\gamma=\frac{R_1}{R_2}>0$. Therefore, the case IV in Sekulic et al. (2011a) should be rejected.

6) The results of Case V in Sekulic et al. (2011a) also do not satisfy the algebraic Equation (21). After a careful revision, we have shown that the correction of this case should be

$$b=-\frac{\gamma}{96}, a_0=\frac{5\gamma}{8\beta},\ b_1=\frac{\alpha\gamma}{80\beta c}, b_2=\frac{-\gamma^2}{1536\beta}, c=\frac{\alpha}{5}\sqrt{\frac{6}{\gamma}},\ a_1=\frac{6\alpha}{5\beta c}, a_2=-\frac{6}{\beta} \qquad 16)$$

It is easy to see that the corrected values of (16) satisfy the algebraic Equation (21) of Sekulic et al. (2011a).

Hence with replacing β by $\frac{\beta}{2}$ in Equation (13), the result (26) of Sekulic et al. (2011a) should be rewritten in the corrected form:

$$\begin{aligned} u(x,t)=&\frac{5\gamma}{4\beta}-\frac{12\alpha}{5\beta c}\sqrt{\frac{\gamma}{96}}\tanh\left(\sqrt{\frac{\gamma}{96}}(\frac{1}{L}x-\frac{\alpha}{5}\sqrt{\frac{6}{\gamma}}\frac{t}{\tau})\right)-\frac{\gamma}{8\beta}\tanh^2\left(\sqrt{\frac{\gamma}{96}}(\frac{1}{L}x-\frac{\alpha}{5}\sqrt{\frac{6}{\gamma}}\frac{t}{\tau})\right)\\ &+\frac{\alpha\gamma}{40\beta c}\sqrt{\frac{96}{\gamma}}\coth\left(\sqrt{\frac{\gamma}{96}}(\frac{1}{L}x-\frac{\alpha}{5}\sqrt{\frac{6}{\gamma}}\frac{t}{\tau})\right)-\frac{\gamma}{8\beta}\coth^2\left(\sqrt{\frac{\gamma}{96}}(\frac{1}{L}x-\frac{\alpha}{5}\sqrt{\frac{6}{\gamma}}\frac{t}{\tau})\right) \end{aligned} \qquad (17)$$

On comparing the corrected result (17) with our result $u_1(x,t)$ of 'Exact traveling wave solutions of Equation (1) for Case 3', we conclude that they are equivalent if $p=0$ and $q=1$.

7) The results of case VI in Sekulic et al. (2011a) do not

also satisfy Equation (21). After a careful revision, we have shown that the correction of this case should be in the form:

$$b = \frac{\gamma}{96},\ c = \frac{\alpha}{5}\sqrt{\frac{-6}{\gamma}},\ b_2 = -\frac{\gamma^2}{1536\beta},\ b_1 = -\frac{\alpha\gamma}{80\beta c},\ a_2 = \frac{-6}{\beta},\ a_1 = \frac{6\alpha}{5\beta c},\ a_0 = \frac{3\gamma}{8\beta} \quad (18)$$

It is easy to see that the corrected values of (18) satisfy the algebraic Equation (21) of Sekulic et al. (2011a). From the values of Equation (18) we deduce that $\gamma = \frac{-6\alpha^2}{25c^2}$ which is negative. This contradicts that $\gamma = \frac{R_1}{R_2} > 0$. Therefore, the case VI in Sekulic et al. (2011) should be rejected.

From these discussions we deduce that our results in the present article are new and recover the well-known results obtained in Sekulic et al. (2011a) after its corrections obtained above.

Conflict of Interests

The author(s) have not declared any conflict of interests.

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