

Full Length Research Paper

Fractional order chaos in Josephson junction

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This paper studies the fractional-order chaotic dynamics of Josephson junction based on resistor and capacitor shunted junction (RCSJ) model proposed by W. C. Stewart and D. E. McCumber. The fractional-order numerical integration is carried out with modified trapezoidal rule. Fractional-order chaotic behaviors of the model are explained by bifurcation diagram. Numerical results confirm that there exists chaos at fractional-order in this model.

Key words: Fractional calculus, Josephson junction, chaotic dynamical system, numerical method.

INTRODUCTION

The theory of fractional calculus is 300 years old mathematical topic. This theory originated from Leibnitz's note in the 17th century. This mathematical theory has long history that dates back to the day that Leibnitz replied query from L'Hospital about the meaning of half order derivative (Oldham and Spanier, 1974). The theory is not known to scientist and engineers that much because there are few solutions for solving the fractional differential equation (Oldham and Spanier, 1974). Recently, the study in this theory has gained more applications such as viscoelasticity, mechanics and nonlinear dynamic and anomalous phenomena (Zaslavsky, 2002). The comprehensive discussions of this theory are presented by Oldham and Spanier (1974), Miller and Ross (1993), and Podlubny (1999) and applications in physics and engineer are presented by Sabatier and Machado (Sabatier et al., 2007).

In general, the non-linear dynamical system can be described by simple rules as well as different physical systems. In certain dynamical system, there exists the irregular behavior that is sensitive to small changes in the

initial condition known as chaos (Strogatz, 1994). Many non-linear systems in nature exhibit irregular pattern or chaotic property by being sensitive to initial condition. Examples include the models that describe the planetary motion, oscillation in electric circuit, swinging of pendulum, the flow of liquid, chemical reaction, fermentation process (Strogatz, 1994), dripping faucet (Dreyer and Hickey, 1991) and population of some species in ecological system (May, 1976).

Chaos is an irregularity behavior which arises in nonlinear dynamical systems. The first discovery of chaos in atmospheric convection model by Lorenz is the beginning of research in this area (Lorenz, 1963). The equations that described the dynamical system are differential equations which yield different type of solutions, such as limit cycle, periodic, periodic doubling, non-periodic and chaotic (Strogatz, 1994).

The fundamental results in fractional order differential equation that are useful for the applications are stable. The studies of fractional-order stability include the fractional Duffing oscillator, fractional predator-prey and

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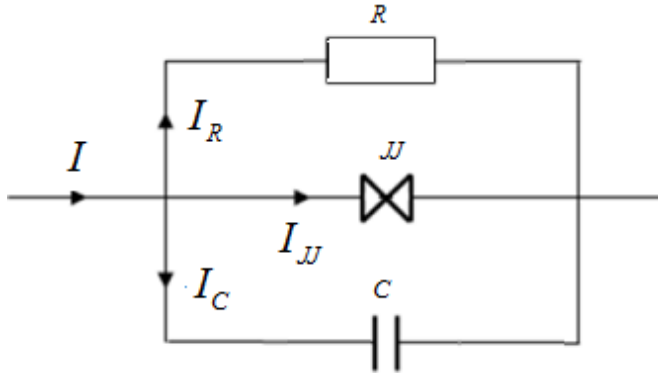


Figure 1. Simple circuit model of Josephson junction.

rabies models. In control theory, the fractional-order control systems have many interesting problems related to stability theory such as robust stability, bounded input – bounded output stability, internal stability, root-locus, robust controllability, robust observability, etc (Li and Zhang, 2011). The stability of fractional order often refers to Matignon's theorem for commensurate order. The theorem analyzes the system stability by locating the eigenvalues of the dynamical system in complex plane. Xiang-Jun et al. (2008) have proven the stability theorem of nonlinear fractional-order differential equation by using the Gronwall-Bellman lemma (Li and Zhang, 2011). Petras provided a survey on the methods for stability investigation of certain class of fractional differential systems with rational orders (Petras, 2009).

The theory of Poincare Bendixson explains that chaos can exist in continuous dynamical system with at least three degrees (Li and Yorke, 1975). The study of nonlinear dynamical systems with theory of fractional calculus provided novel conclusion; for example, chaotic dynamics of fractional-order Arneodo's systems (Lu, 2005), fractional Chen system (Lu et al., 2006), chaos in the Newton–Leipnik system with fractional order (Wang and Tian, 2006), chaos in a fractional order modified Duffing system (Ge and Ou, 2007), fractional order Chua's system (Radwan et al., 2011; Hartley et al., 1995; Petras, 2008), fractional-order Volta's system (Petras, 2009), chaos in a fractional-order Rössler system (Li and Chen, 2004, Zhang et al., 2009), fractional-order Lorenz chaos (Yu et al., 2009).

Nonlinear physical phenomena are often described by integer order differential equations. With the development of fractional calculus, it has been found that the behavior of many systems can be described by using the fractional differential systems fractional order damping of duffing system (Cao et al., 2010). Josephson junction is a nonlinear device that has many applications in high frequency, ultra-low noise and low power consumption (Chen et al., 2012). In this paper, we study further the fractional dynamic of Josephson junction circuit from the well-known. The Resistively and Capacitively Shunted

Junction-model (RCSJ) proposed by Stewart McCumber. This device is a good candidate for studying the fractional-order chaotic patterns. The chaotic behaviors of fractional order have been investigated by the bifurcation diagram and phase space. Numerical algorithm was carried out using modified trapezoidal rule proposed by Odibat and Momani, (2008b). The aim of this study is to examine the intermediate state of the bifurcation patterns.

METHODOLOGY

Mathematical model of Josephson junction

A simple model of the Josephson junction (JJ) can be considered with a resistor and capacitor shunted junction (RCSJ) circuit as represented in Figure 1. The model was proposed by W. C. Stewart and D. E. McCumber (Clarke and Braginski, 2006). The current through the circuit is determined with the Kirchoffs law as follows (Chen et al., 2012):

$$I = I_{JJ} + I_C + I_R = I_c \sin \varphi + C \frac{dV}{dt} + \frac{V}{R} \quad (1)$$

A standard form of the RCLSJ model is proposed as:

$$\frac{h}{4\pi e} \frac{d\varphi}{dt} = V, \quad (2)$$

$$C \frac{dV}{dt} + \frac{V}{R_s} + I_c \sin(\varphi) + I_s = I, \quad (3)$$

$$L \frac{dI_s}{dt} + I_s R_s = V, \quad (4)$$

Let $x = \varphi, y = \frac{V}{I_c R_s}, z = \frac{I_s}{I_c}$ and

$$\beta_c = 4\pi I_c R_s^2 C / h, \beta_L = 4\pi e I_c L / h, g = R_s / R_v$$

where β_c and β_L are simplified capacitance and inductance. We can rewrite the above relation into three differential equations as follows (Feng and Shen, 2008):

$$x' = y \quad (5)$$

$$y' = \frac{1}{\beta_c} (i - gy - \sin(x) - z) \quad (6)$$

$$z' = \frac{1}{\beta_L} (y - z) \quad (7)$$

By replacing the derivative with fractional derivative, we have the fractional-order differential equations of RCSJ model as follows:

$$D^q x = y \quad (8)$$

$$D^q y = \frac{1}{\beta_c} (i - gy - \sin(x) - z) \tag{9}$$

$$D^q z = \frac{1}{\beta_L} (y - z) \tag{10}$$

Normalized time $\tau = \omega_c t$, $\omega_c = 4\pi I_c R_s / h$

Normalized current $i = I / I_c$

Normalized voltage $v = V / I_c R_s$ (Feng and Shen, 2008).

Fractional calculus

In fractional calculus, there are many definitions proposed. Two mostly use definitions are Riemann-Liouville and Grunwald-Letnikov. The definition that is suitable for studying the analytic solution is Riemann-Liouville definition while the definition that is more appropriate for numerical calculation is Grunwald-Letnikov definition (Podlubny, 1999).

Definition 1: The Riemann-Liouville fractional integral of order $q > 0$ of a function $f : R^+ \rightarrow R$ is given by Podlubny (1999):

$$I^q f(x) = \frac{1}{\Gamma(q)} \int_0^x (x-t)^{q-1} f(t) dt \tag{11}$$

Podlubny (1999) provided the right side point-wise defined on R^+ where $\Gamma(\cdot)$ is Gamma function defined by:

$$\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} du \tag{12}$$

Definition 2: The Caputo fractional derivative of order $q \in (n-1, n)$ of a continuous function $f : R^+ \rightarrow R$ is given by Podlubny (1999):

$$D^q f(x) = I^{n-q} D^n f(x), D = \frac{d}{dt} \tag{13}$$

Definition 3: Grunwald-Letnikov definition for fractional derivative of order q is given by Podlubny (1999):

$${}_a D_t^q f(t) = \lim_{h \rightarrow 0} h^{-q} \sum_{j=0}^{\lfloor (t-a)/h \rfloor} (-1) \binom{q}{j} f(t - jh) \tag{14}$$

Where,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$= \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)} \tag{15}$$

In classical calculus, the meaning of integer order derivative is the rate of change, direction of decline or slope in the geometric interpretation. The meaning of fractional-order derivative is different. It has no conventional geometric meaning and physical interpretation of fractional integration and differentiation. Recently, Podlubny has proposed a new physical interpretation based on general convolution integrals of the Volterra type (Podlubny, 1999). Machado gives a geometric and probabilistic interpretation based on Grunwald-Letnikov definition of the fractional derivative (Machado et al., 2009). Fractional derivative and integration have many forms and definitions. The operators in this mathematical theory are fascinating. The fractional derivative of the constant value is not zero (Podlubny, 1999). For three centuries, the theory of fractional derivatives is developed only for mathematicians.

There are many approaches for solving the fractional order differential equation which include numerical approximation, series expansion, method of decomposition, predictor corrector scheme (Petras, 2008), Adam Moulton algorithm, Laplace transform (Tavazoei et al., 2008), differential transform (Odibat et al., 2010), (Odibat and Momani, 2008a), (Odibat et al., 2008c) and numerical calculation of nonlinear fractional partial differential equations (Momani et al., 2008). Solutions are written in the compact form of mathematical function mathematicalfunction (Erturk et al. 2012; Erturk et al., 2008; Baleanuet al., 2009; Li and Deng, 2007). The applications of fractional calculus in physics are better for describing the diffusion phenomena in homogeneous media with non-integer derivative; the fractional derivatives model of viscoelastic material (Bagley and Calico, 1991); fractional order impedance in electric circuit; dynamical process of heat conduction and chaotic dynamical system (Podlubny, 1999).

The advantage of fractional derivatives compared to classical integer-order calculus is the description of memory and hereditary properties of various materials and processes. In the last few decades, many authors show that derivatives and integral of non-integer order are very suitable for describing properties of various real materials e.g. polymers. It has been shown that fractional-order models are more adequate than integer-order models (Bagley and Torvik, 1983). Recently, this mathematical theory has gained more attention due to their vast applications in Applied Sciences (Podlubny, 1999).

Numerical method

There are many approaches in fractional order numerical calculation. Two main approaches for numerical calculation are the frequency domain and time domain. From the study of Tavazoei, the frequency domain approach can lead to the fake chaotic results (Tavazoei et al., 2008). The numerical method that we utilized in this paper is the modified trapezoidal rule proposed by Odibat and Momani, (2008b). This method is simple calculation scheme that was derived from the area of trapezoidal shape.

Consider $y = f(x)$ over $[a, b]$ suppose that the interval $[a, b]$ is subdivided into m subintervals $\{[x_{k-1}, x_k]\}_{k=1}^m$ of equal width $h = \frac{b-a}{m}$ by using the equally spaced nodes $x_k = x_0 + kh$ for $k = 1, 2, \dots, m$.

The composite trapezoidal rule for m subinterval is:

$$T(f, h) = \frac{h}{2} (f(a) + f(b)) + h \sum_{k=1}^m f(x_k) \tag{16}$$

The formula can be extended by using fractional order differential as follows (Odibat and Momani, 2008b):

$$T(f, h, q) = ((k-1)^{q+1} - (k-q-1)k^q) \frac{h^q f(0)}{\Gamma(q+2)} + \sum_{j=1}^{k-1} ((k-j+1)^{q+1} - 2(k-j)^{q+1} + (k-j-1)^{q+1}) \frac{h^q f(x_j)}{\Gamma(q+2)} \tag{17}$$

The formula in Equation (17) is used to approximate the integral function of arbitrary order. According to the classical theory of ordinary differential equations, we need to specify initial conditions to produce a unique solution for the problem (Odibat and Momani, 2008b).

Equation (18) is an approximation to fractional integral at order q :

$$(I^q f(x)(a)) = T(f, h, q) - E(f, h, q) \tag{18}$$

where $a > 0, q > 0$

Odibat and Momani (2008b) shows that the error is a function of parameter h

$$|E(f, h, q)| = O(h^2) \tag{19}$$

It is obviously that if the order $q = 1$, then the modified trapezoidal rule is reduced to the classical trapezoidal rule:

$$\lim_{q \rightarrow 1} D^q f(x) = \frac{df(x)}{dx} \tag{20}$$

Stability of fractional-order chaos

Here we discuss the stability condition for fractional-order systems and a necessary condition for chaos to exist. Consider the following fractional differential equations (Petras, 2009; Podlubny, 1999; Li et al., 2011; Odibat et al., 2008c).

$$D^q x_i = f_i(x_1, x_2, x_3), \quad i = 1, 2, 3 \tag{21}$$

The equilibrium points x_i^{eq} of the fractional-order differential system can be obtained by solving the following equations:

$$D^q x_i = 0 \tag{22}$$

and δ_i is a small deviation from the equilibrium.

$$f_i(x_1^{eq} + \delta_1(t), x_2^{eq} + \delta_2(t), x_3^{eq} + \delta_3(t)) = f_i(x_1^{eq}, x_2^{eq}, x_3^{eq}) + \left. \frac{\partial f_i}{\partial x_1} \right|_{eq} \delta_1 + \left. \frac{\partial f_i}{\partial x_2} \right|_{eq} \delta_2 + \left. \frac{\partial f_i}{\partial x_3} \right|_{eq} \delta_3 \tag{23}$$

Then

$$D^q (\delta_i) \cong \left. \frac{\partial f_i}{\partial x_1} \right|_{eq} \delta_1 + \left. \frac{\partial f_i}{\partial x_2} \right|_{eq} \delta_2 + \left. \frac{\partial f_i}{\partial x_3} \right|_{eq} \delta_3 \tag{24}$$

Therefore, we will have a linear system:

$$D^q (\delta_i) = A \delta \tag{25}$$

Where:

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix} \tag{26}$$

The following autonomous system is:

$$D^q x = Ax, x(0) = x_0 \tag{27}$$

where $0 < q < 1, x \in R^n, A \in R^{n \times n}$ is asymptotically stable if and only if $|\arg(\lambda_i(A))| > q\pi/2$. In this case, each component of the states decays towards 0 like t^{-q} . Also, this system is stable if and only if $|\arg(\lambda_i(A))| \geq q\pi/2$ and those critical eigenvalues that satisfy $|\arg(\lambda_i(A))| = q\pi/2$ have geometric multiplicity one (Matignon, 1996).

Given the parameters as follows $i = 1.11, \beta_c = 0.707, \beta_L = 2.68, g = 0.0478$, then we can have the asymptotically stable condition $q < \frac{2}{\pi}(2.53892) \approx 1.62$.

NUMERICAL RESULTS

Lyapunov exponent

Nonlinear dynamical system is often defined by the differential equation. The dynamical system may have attracting limit set. The different attracting limit set with different basin of attraction is determined by the initial condition. The limit set can be characterized by four fundamental types of limit sets: fixed points, periodic, quasi-periodic and chaos. The dynamical system can be used to measure the rate of divergence of nearby trajectories using the Lyapunov exponents. These quantities can characterize the solutions of the differential equations (Sandri, 1996). The Lyapunov exponents of the Josephson junction model for $i = 1.11, i = 1.56$ and $\beta_c = 0.707, \beta_L = 2.68, g = 0.0478$ are shown in Figure 2.

Fractional-order phase space

The study of dynamical system behavior entails studying the behavior of the trajectories in the phase space. The

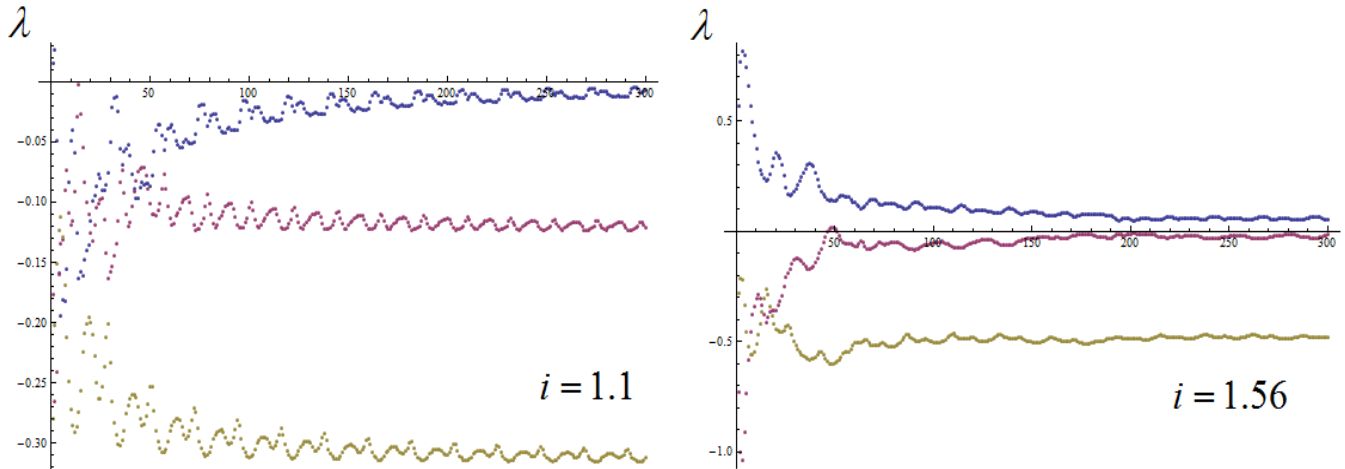


Figure 2. The Lyapunov spectrum for the Josephson junction model.

phase space in the structural form of trajectories is examined. The structures of the trajectories are diagram of phase space of the dynamical systems. From a geometrical point of view, this structure is the geometrical pattern of the relative positions of trajectories in the phase space. The study of phase space by fractional calculus helps in understanding the small deviation of the dynamical systems. The behavior of chaotic dynamical system can be understood by construction viewed from a phase space perspective. In Josephson junction circuit, the chaotic behavior is often viewed from the phase space of the voltage and current of the dimensionless variables. The geometric meaning of the single close loop of phase space presents the periodic system and two close loops of phase space represent the periodic doubling and so on. The chaotic system of the phase space appears as a messy loop.

The numerical results of fractional-order phase space of the variable y and z are shown in Figure 3. According to the numerical results, the change in integration-order results as bent curve or distorted trajectories.

Memory effect with Bifurcation theory

The behaviors of nonlinear system are intriguing because they range from simple to complex and periodic to non-periodic. In certain nonlinear system, chaotic behavior is exhibited at certain range. Two different nonlinear systems can be described by similar set of equations. The study of sensitivity in nonlinear dynamical system helps in understanding how systems behave at certain parameters. The behaviors of nonlinear dynamical system are studied by bifurcation theory. Bifurcation theory is mathematical theory that study changes in qualitative or topological structure of the dynamical

system. The theory explains the behavior of the integral curves of the vector fields and the solution of differential equation (Strogatz 1994). The combination of fractional calculus and this mathematical theory help us in understanding how system change upon the order.

Integer-order Bifurcation diagram

Bifurcation pattern can be obtained by varying certain parameters. In this case we use β_L . In the diagrams (Figure 4), the values of i vary from 1.12 to 1.15 results in bifurcation diagram (Figure 3). Numerical results show different bifurcation patterns start from periodic and become chaotic; they disappear when β_L increases. The dot bands are the chaotic regions. There is periodic doubling and quadrupling of the value of $i = 1.14$; it disappears when $i = 1.15$. According to the pattern, we may conclude that different values of parameters provide different bifurcation patterns.

Fractional order Bifurcation diagram

The numerical integration of fractional order is calculated by applying the modified trapezoidal rule to the differential equation of the RCSJ model. In order to reduce numerical errors, the time-step of the fractional-order numerical integration is set to $h = 2000$. The diagrams are obtained by varying the fractional integration order with the fixed values of the parameters. In this calculation, we use $i = 1.59, i = 1.11$; while the other variables are $\beta_c = 0.707, \beta_L = 2.68, g = 0.0478$.

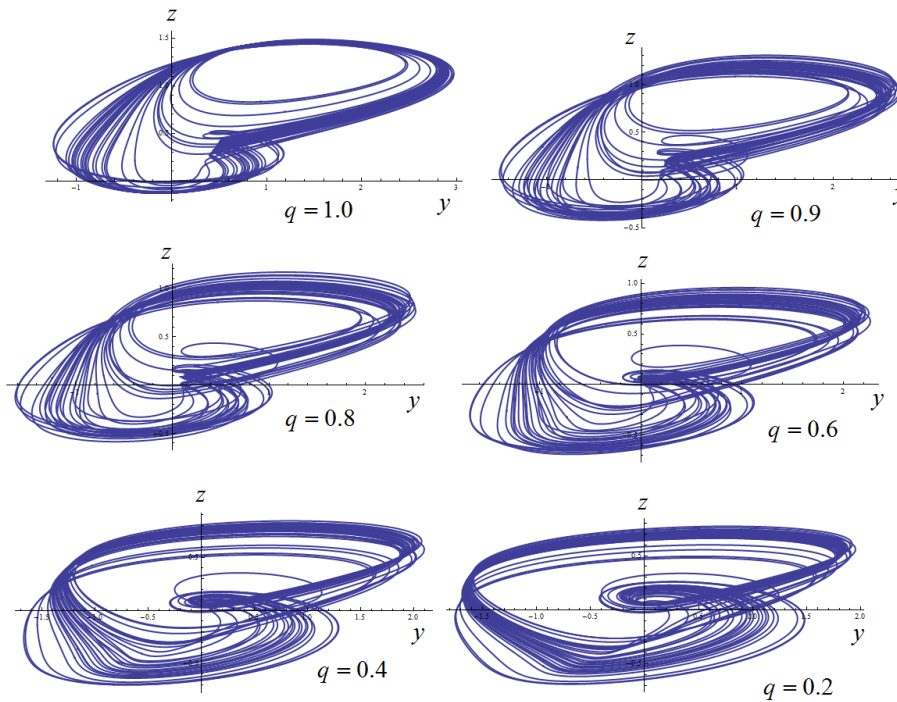


Figure 3. Fractional-order phase space of RCSJ model for following parameters $i = 1.11, \beta_c = 0.707, \beta_L = 2.68, g = 0.0478$.

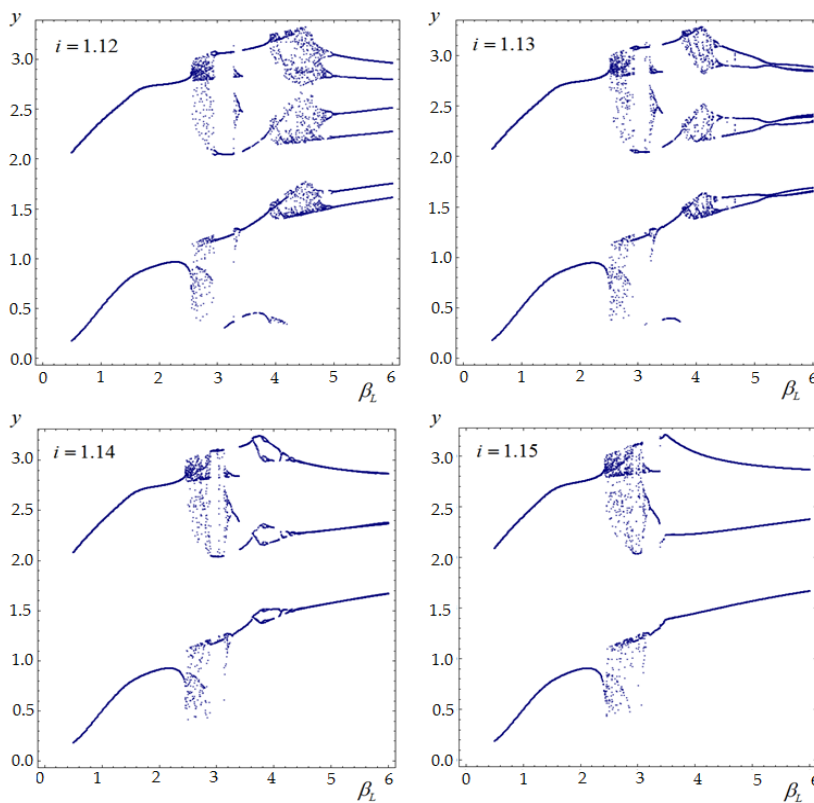


Figure 4. Integer-order Bifurcation diagram for the following parameters $i = 1.12 - 1.15, \beta_c = 0.707, \beta_L = 2.68, g = 0.0478$.

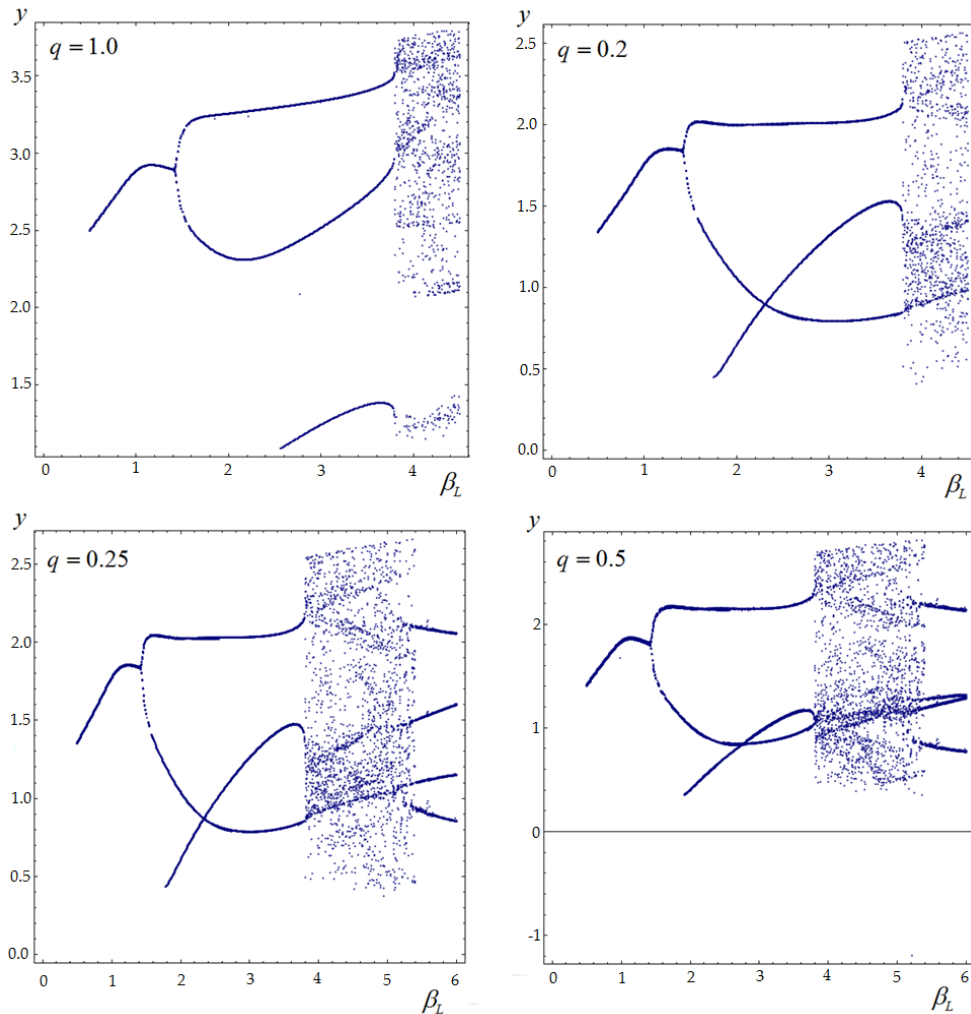


Figure 5. Fractional-order Bifurcation diagram of the following parameter $i = 1.59, \beta_c = 0.707, \beta_L = 2.68, g = 0.0478$.

For the values $i = 1.59$, the chaotic region begins with $\beta_L \approx 4$. For the values $i = 1.11$, the chaotic region begins with $\beta_L \approx 3.6$. The results are as in Figure 5 and Figure 6. We may recognize the existing pattern in fractional-order bifurcation diagram where the structures of each diagram resemble the previous stage.

DISCUSSION

The combination between theory of fractional calculus and nonlinear dynamical system provides the novel idea for exploring the unexplored region. Even simple known chaotic systems have broad ranges of irregular behavior that are not yet examined. Fractional calculus helps in exploring the unexplored areas in dynamical system. The RCSJ model is one of the simple known systems that

there exhibit chaotic behavior at certain ranges. Theory of fractional differential equations has proved to be a valuable tool for the modeling of many physical phenomena. In this paper, we have applied theory of fractional calculus to examine the fractional-order bifurcation diagram in The Resistively and Capacitively Shunted Junction-model (RCSJ) model. We may see that physical system does not only depend on the instant of time but also on the history of the earlier stage, which can be achieved by numerical calculation with fractional calculus. The numerical results show that the patterns of the fractional-order bifurcation diagram do not differ from the integer order that much. We may conclude that integer-order bifurcation diagram explains the behaviors of dynamics subject to change in parameters. The results in fractional order bifurcation diagram are the deviations from the previous stages since the fractional operator has memory effect.

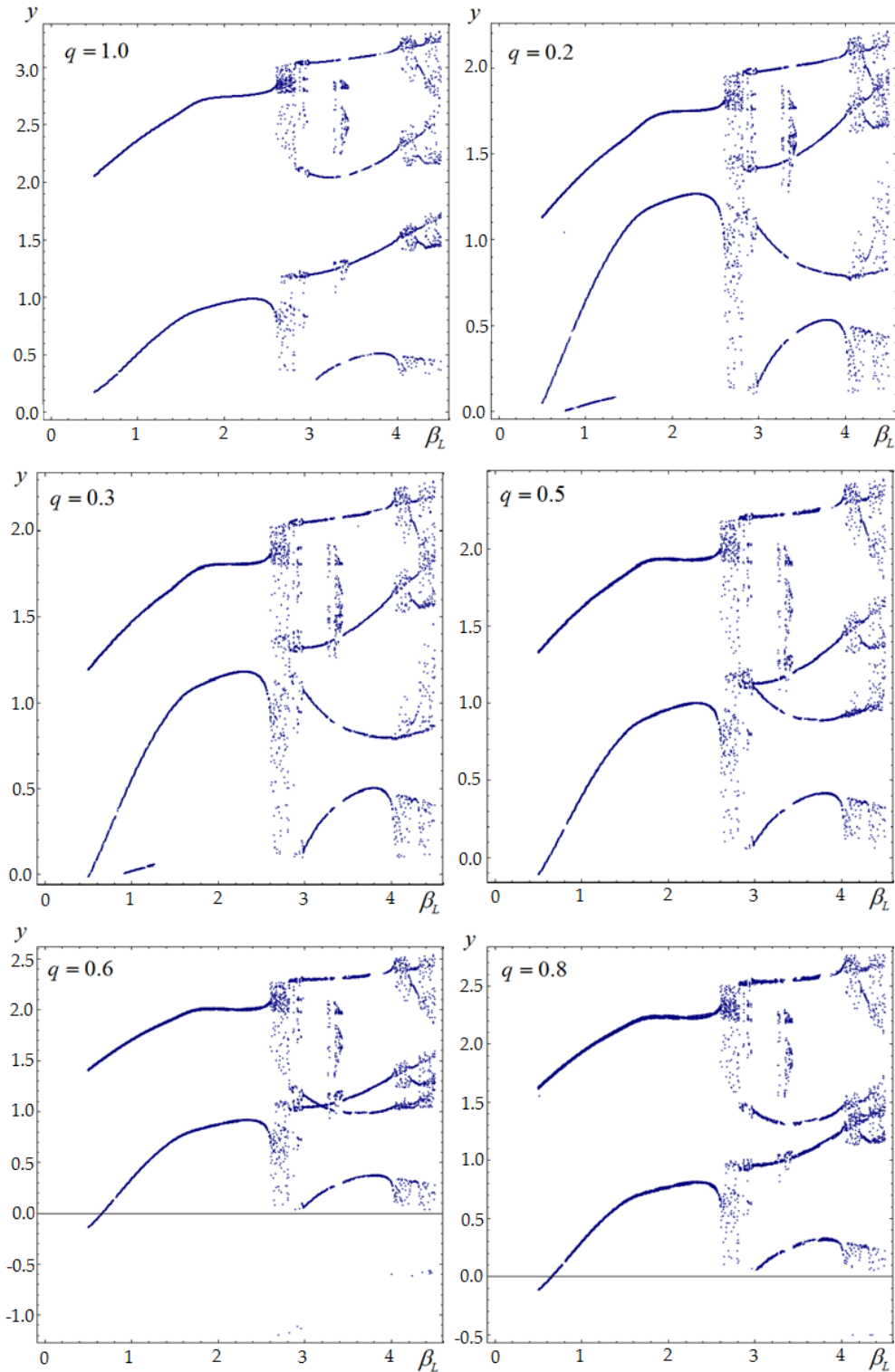


Figure 6. Fractional-order Bifurcation diagram of the following parameter $i = 1.11, \beta_c = 0.707, \beta_L = 2.68, g = 0.0478$.

Conflict of Interests

The author(s) have not declared any conflict of interests.

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