

Full Length Research Paper

Variational cartographic projections for the Slovak Republic territory

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Disadvantages of currently used Křovák's projection in Slovak Republic such as large scale distortion became evident after dividing of Czechoslovakia. The new proposal of the cartographic projection for the Slovak Republic (Lambert's conformal conic projection in normal position) was created in 2010. The aim of this paper is to define an optimal oblique position of map projections for the Slovak Republic territory and to design two cartographic projections in different positions (normal, oblique) using variational criteria modified for conformal projections for the Slovak Republic. Used calculus, the achieved scale distortions and their comparison with scale distortions of currently used map projections will be demonstrated.

Key words: Conformal cartographic projection, variational criteria, scale distortion, variational cartographic projection.

INTRODUCTION

The currently used cartographic projection in the Slovak Republic (and also in the Czech Republic) is the Křovák's projection which is an oblique case of a conformal conic projection based on two preserved parallels and was designed in 1922 by Josef Křovák strictly for Czechoslovakia. The Bessel's reference ellipsoid is transformed to a sphere using Gauss' conformal projection. Then, the sphere is transformed to a secant cone in oblique position. The coordinates are positive to the south and west (Srňka, 1986). Disadvantages of this projection such as large scale distortion became evident after dividing of Czechoslovakia. In 2010 a new

projection, Lambert's projection with parameters for Slovakia was designed at the request of the Geodesy, Cartography and Cadastre Authority of the Slovak Republic by Margita Vajsáblová. This new conformal projection uses a normal cone, secant to the reference ellipsoid GRS 1980 and was designed for the Slovak Republic territory (Vajsáblová, 2011). The reference ellipsoid is transformed directly on a secant cone in normal position. The coordinates are positive to the north and east.

One of the basic problems in cartography is the problem of optimal map projections, especially in the area

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of the creation of large-scale maps and partially medium-scale maps, that is, those projections of a given domain on a reference surface onto the plane under which the distortion is minimal. The most important factors of selection of map projection are shape, size and position of projected territory, also purpose and scaling factor of the map. The aim of this paper is to design two map projections in different positions (normal, oblique) using variational criteria modified for conformal projections for the Slovak Republic. Examples of using these criteria to derive and optimize map projections for other territories (Canada, Czech Republic) and their valuation and comparison are described in Frankich (1982), Bořík (1999) and Hojovec et al. (1975).

The optimal oblique position of cartographic projection for the Slovak Republic territory is defined in this paper and the aim is to design two cartographic projections in different positions (normal, oblique) using variational criteria modified for conformal projections for the Slovak Republic. Used calculus, the achieved scale distortions and their comparison with scale distortions of currently used map projections will be demonstrated.

Valuation of cartographic projections

The type of criteria for deciding the merits of map projections are as follow (Hojovec et al., 1987):

- a) Extremal and minimaximal,
- b) Variational (additive, integral) criteria.

In *extremal criteria* we evaluate conformal projection by:

- i. Maximal value of scale distortion: $|m - 1|_{max}$,
- ii. Maximal value of area distortion: $|m_{pl} - 1|_{max}$.

The equivalent projections are evaluated by maximal value of angle distortion: $|\Delta\omega|_{max}$.

Chebyshev's theorem for conformal projection, in which natural logarithm of scale distortion is the least different from zero, if scale distortion of closed boundary curve is constant (Chebyshev, 1962). These projections are called minimaximal type projections.

Meshcheryakov's theorem for Euler's type projections (orthogonal equivalent): the best of Euler's projection by minimaximal criterion is the projection in which image of the central meridian is a line and the scale distortion in every points holds, that scale distortion of meridian and parallel are equal to 1 (Meshcheryakov, 1968).

The most popularized variational type criteria for valuation of map projections are the following:

- i. The Airy's criterion (1861), where h^2 is mean quadratic scale distortion defined by extremal scale factors m_a, m_b :

$$h^2 = \frac{1}{2} [(m_a - 1)^2 + (m_b - 1)^2], \tag{1}$$

- ii. The Jordan's criterion, which characterizing scale distortion in a point by h^2 for all the azimuths A:

$$h^2 = \frac{1}{2\pi} \int_0^{2\pi} (m_A - 1)^2 dA, \tag{2}$$

in some case for several value of the azimuth A:

$$h^2 = \frac{1}{\mu} \sum (m_A - 1)^2, \tag{3}$$

where m_A is factor of scale distortion appertaining to direction with azimuth A, μ is number of these directions.

- iii. The Kavraiskii's criteria are modification of previous criteria by using the logarithmic values (Kavraiskii, 1959):
- the Airy-Kavraiskii criterion:

$$h^2 = \frac{1}{2} [\ln^2 m_a + \ln^2 m_b], \tag{4}$$

- the Jordan-Kavraiskii criterion:

$$h^2 = \frac{1}{2\pi} \int_0^{2\pi} \ln^2 m_A dA, \tag{5}$$

$$h^2 = \frac{1}{\mu} \sum \ln^2 m_A. \tag{6}$$

Characteristic value of cartographic projection of domain Δ of reference surfaces is value of integral I:

$$I = \frac{1}{\Delta} \iint_{\Delta} h^2 \cos U dU dV, \tag{7}$$

or for n choices points:

$$I = \frac{1}{n} \sum h^2. \tag{8}$$

Mentioned criteria are possible to use in deriving of cartographic projection too, thereby we obtain variational type of projection (Urmaev, 1947).

The conformal projections are frequently applied in coordinate systems, because in this type of projection holds, that scale distortion is independent to azimuth, we

are able to simplify previous formulae in the next chapters.

Variational cartographic projections

We are concerned with creation of conformal map projections of territory with minimal scale distortion by variational Criterion (7), where we apply the Airy-Kavraiskii Criterion (4) modified for conformal projection ($m_a = m_b = m$):

$$h^2 = \ln^2 m. \tag{9}$$

Projected territory is situated on reference sphere between parallels with geographic latitude U_S, U_N and between meridians with geographic longitude V_W and V_E , where we use their relative value from central meridian V_0 :

$$V_0 = \frac{V_W + V_E}{2}. \tag{10}$$

In consideration of symmetry of spherical trapezium and after substitution of (9) to variational Criterion (7) we obtain:

$$I = \frac{1}{\Delta} \int_0^{V_1} \int_{U_S}^{U_N} \ln^2 m \cos U \, dU \, dV, \tag{11}$$

where V_1 is maximum of relative geographic longitude. We need to find function $\ln m$ from condition that value of I is minimal.

In conformal projection it holds (Hojovec et al. 1975):

$$\frac{\partial^2 u}{\partial Q^2} + \frac{\partial^2 u}{\partial V^2} = 0, \tag{12}$$

where:

$$u = \ln m \cos U \tag{13}$$

and Q is isometric latitude calculated by:

$$Q = \ln \operatorname{tg} \left(\frac{U}{2} + 45^\circ \right). \tag{14}$$

Solution of Equation (12) has a shape (Hojovec et al., 1975):

$$u = \sum_{j=0}^n (a_j \psi_j + b_j \tau_j), \tag{15}$$

where ψ_j and τ_j were determine by:

$$\psi_j + i\tau_j = (Q + iV)^j. \tag{16}$$

Two complex variables are equal than its imaginary and real components are equal, for ψ_j and τ_j we obtain ($j = 0$ to $j = 4$):

$$\begin{aligned} \psi_0 = 1, & \quad \psi_1 = Q, & \quad \psi_2 = Q^2 - V^2, & \quad \psi_3 = Q^3 - 3QV^2, & \quad (17) \\ \psi_4 = Q^4 - 6Q^2V^2 + V^4; & \quad \tau_0 = 0, & \quad \tau_1 = V, & \quad \tau_2 = 2QV, \\ \tau_3 = 3Q^2V - V^3, & & \quad \tau_4 = 4Q^3V - 4QV^3. & & \end{aligned}$$

Equation (15) is simplified for symmetric:

$$u = \sum_{j=0}^n a_j \psi_j. \tag{18}$$

After substitution Equations (13) and (18) to Airy-Kavraiskii criterion (11) we have:

$$I = \frac{1}{\Delta} \int_0^{V_1} \int_{U_S}^{U_N} \left(\sum_{j=0}^4 a_j \psi_j - \ln \cos U \right)^2 \cos U \, dU \, dV. \tag{19}$$

By minimalisation of (19) we obtain system of five equations for $j = 0$ to 4, where j is:

$$a_0 P_{j0} + a_1 P_{j1} + a_2 P_{j2} + a_3 P_{j3} + a_4 P_{j4} = F_j, \tag{20}$$

where functions P and F hold (Hojovec et al. 1975):

$$P_{jk} = \int_0^{V_1} \int_{U_S}^{U_N} \psi_j \psi_k \cos U \, dU \, dV, \quad F_k = \int_0^{V_1} \int_{U_S}^{U_N} \psi_k \cos U \ln \cos U \, dU \, dV. \tag{21}$$

The coefficients a_j are calculated for domain defined by marginal parallels U_S, U_N and meridians $0^\circ, V_1$. The factor of scale distortion is calculated by:

$$\ln m = \sum_{j=0}^4 a_j \psi_j - \ln \cos U. \tag{22}$$

Proposal of two variational cartographic projections for the Slovak Republic

The geographic coordinates φ, λ of the boundary points of the Slovak Republic on reference ellipsoid GRS 1980,

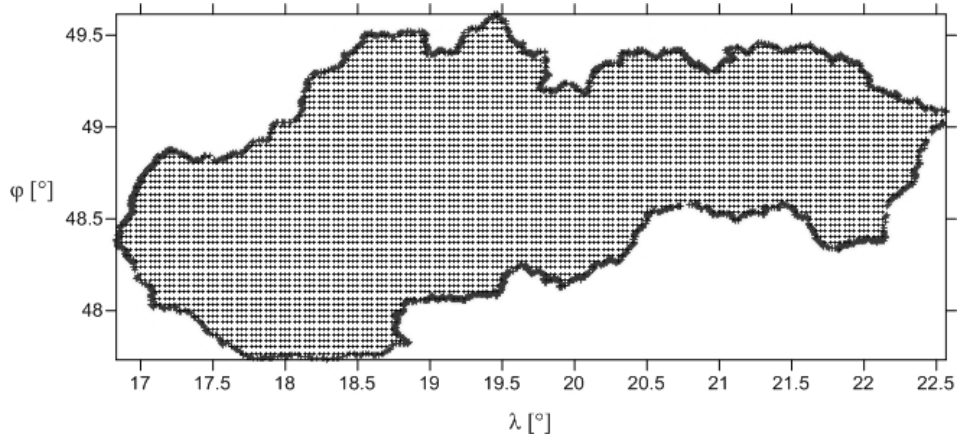


Figure 1. Test points for calculating the factor of scale distortion.

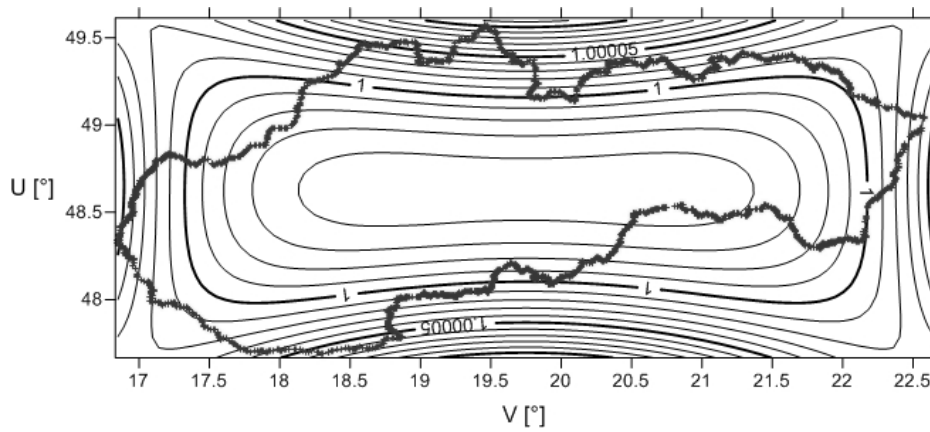


Figure 2. Curves of constant scale distortion – normal case of variational cartographic projection.

we have transformed on the spherical coordinates U, V by Gauss` conformal projection:

$$\operatorname{tg}\left(\frac{U}{2}+45^{\circ}\right)=k\left[\operatorname{tg}\left(\frac{U}{2}+45^{\circ}\right)\sqrt{\left(\frac{1-e\sin\varphi}{1+e\sin\varphi}\right)^c}\right]^{\alpha}, \quad V=\alpha\cdot\lambda. \quad (23)$$

with parameters: $\alpha = 1.000\ 640\ 596\ 750\ 5$ and $k = 1.003\ 315\ 995\ 637\ 3$.

Preserved parallel has geographic latitude $\varphi_0 = 48^{\circ}\ 40'\ 21.2520''$ and radius of the sphere is $R = 6\ 380\ 840.721\ 3\ \text{m}$.

The Slovak Republic is situated on this sphere between parallels with geographic latitude $U_S = 47^{\circ}\ 41'\ 27.0720''$, $U_N = 49^{\circ}\ 34'\ 15.7881''$ and between meridians with geographic longitude $V_W = 16^{\circ}\ 50'\ 32.3600''$ and $V_E = 22^{\circ}\ 34'\ 50.6852''$, where we have used their relative

values from central meridian V_0 calculated as the arithmetic mean (10).

Values of evaluated coefficients a_j for the first domain defined by marginal parallels U_S, U_N and meridians $0^{\circ}, V_1$ and with using Equations (11) to (21) are:

$$a_0 = +17.334734, \quad a_1 = -70.610239, \quad a_2 = +107.550822, \\ a_3 = -73.595801, \quad a_4 = +18.886817.$$

All of the calculus has been realized by system *Wolfram Mathematica 8*. We have calculated the factor of scale distortion for 5 424 points (Figure 1) by Equation (22), the curves of constant scale distortion have been visualized in *Surfer 9* (Figure 2).

The second variational projection is proposed in oblique position, therefore on domain defined on reference sphere between two cartographic parallels with

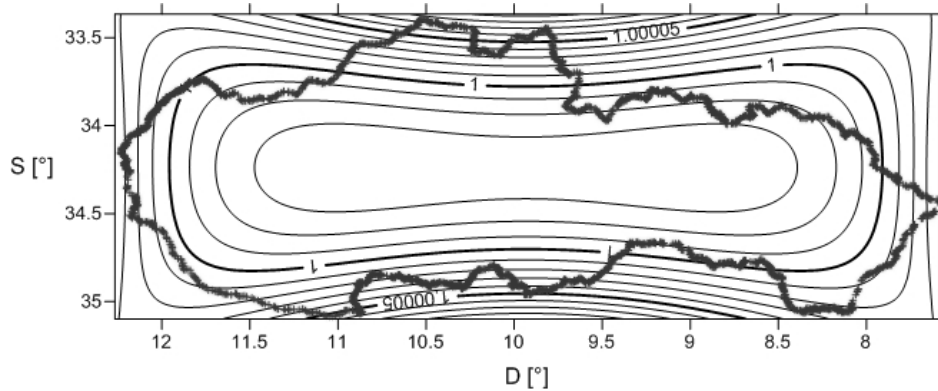


Figure 3. Curves of constant scale distortion – oblique case of variational cartographic projection.

cartographic latitude S_S, S_N and two cartographic meridians with cartographic longitude D_W, D_E . We have defined the values of this cartographic coordinates by consecutive steps:

1st We have chosen three points A, B, C on the south boundary of the Slovak Republic with geographic coordinates:

- $A [U_1 = 48^\circ 19' 49.9601'', V_1 = 22^\circ 05' 58.6244'']$,
- $B [U_2 = 48^\circ 17' 28.5876'', V_2 = 21^\circ 49' 51.4052'']$,
- $C [U_3 = 47^\circ 42' 47.2572'', V_3 = 18^\circ 30' 50.1205'']$.

2nd We have approximated the cartographic parallel circle through the points A, B, C and have calculated the geographic coordinates U_K and V_K of cartographic pole K by equation:

$$\begin{aligned} \operatorname{tg} V_k &= \frac{(\cos U_1 \cos V_1 - \cos U_3 \cos V_3)(\sin U_1 - \sin U_2) - \dots}{(\cos U_1 \sin V_1 - \cos U_2 \sin V_2)(\sin U_1 - \sin U_3) - \dots} \\ &\dots \frac{-(\cos U_1 \cos V_1 - \cos U_2 \cos V_2)(\sin U_1 - \sin U_3)}{-(\cos U_1 \sin V_1 - \cos U_3 \sin V_3)(\sin U_1 - \sin U_2)}, \quad (24) \\ \operatorname{tg} U_k &= \frac{\cos U_2 \cos(V_k - V_2) - \cos U_1 \cos(V_k - V_1)}{\sin U_1 - \sin U_2}. \end{aligned}$$

from that $U_K = -5^\circ 53' 41.1964''$ and $V_K = 32^\circ 08' 18.5219''$.

3rd Then we have transformed geographic coordinates U, V of the points of Slovakia boundary on cartographic coordinates S, D . Extremal values of cartographic latitudes S_S, S_N and cartographic longitudes D_W, D_E are the coordinates of marginal parallels and marginal meridians of boundary trapezium. The Slovak Republic is situated on the sphere between parallels with

cartographic latitude $S_S = 35^\circ 05' 05.7509''$, $S_N = 33^\circ 23' 42.1307''$ and between meridians with cartographic longitude $D_W = 12^\circ 13' 59.6088''$, $D_E = 7^\circ 34' 55.6498''$, where we have used their relative values from central meridian D_0 calculated as the arithmetic mean:

$$D_0 = \frac{D_W + D_E}{2}, \quad (25)$$

from where the maximal value of relative cartographic longitude is D_1 . The interval of latitudes is $1^\circ 52' 48.7161''$ in normal position, $1^\circ 41' 23.6202''$ in oblique position.

The coefficients a_j have been evaluated for the second domain defined by marginal parallels S_S, S_N and meridians $0^\circ, D_1$ by Equations (11) to (21) where the following substitutions have been used:

$$U = S, \quad V = D, \quad U_S = S_S, \quad U_N = S_N, \quad V_1 = D_1. \quad (26)$$

Results of coefficients a_j :

$$\begin{aligned} a_0 &= +6.800198, \quad a_1 = -42.196398, \quad a_2 = +98.029304, \\ a_3 &= -102.608260, \quad a_4 = +40.284538. \end{aligned}$$

All the calculus have been realized by system *Wolfram Mathematica 8*. We have calculated the factor of scale distortion for 5 424 points (Figure 1) by Equation (22) where $U = S$, the curves of constant scale distortion have been visualized in *Surfer 9* (Figure 3).

We have compared these variational cartographic projections with the currently used projection defined by the law – Křivák's projection (Srňka, 1986). The Křivák's projection is an oblique case of conformal conic projection based on two preserved parallels designed in

Table 1. Extremal scale distortions in cartographic projections.

Cartographic projection	Scale distortion [cm/km]
Variational – normal	From -4.6 to +9.7
Variational – oblique	From -3.8 to +7.2
Křovák	From -10.0 to +11.0
Lambert	From -6.7 to +6.7

Table 2. Scale distortions and their quadratic values in cartographic projections by additive criterion.

Cartographic projection	Additive criterion	
	cm ² /km ²	\sqrt{I} [cm/km]
Variational – normal	8.667	2.944
Variational – oblique	6.076	2.465
Křovák	51.125	7.150
Lambert	25.489	5.049

Table 3. Scale distortions and their quadratic values in cartographic projections by integral criterion.

Cartographic projection	Integral criterion	
	cm ² /km ²	\sqrt{I} [cm/km]
Variational – normal	8.709	2.951
Variational – oblique	6.131	2.476
Křovák	48.375	6.955
Lambert	26.211	5.120

1922 by Josef Křovák. Used in the Czech Republic and Slovakia. In 2010 a new projection, Lambert's projection with parameters for Slovakia (Vajsáblová, 2011) was designed at the request of the Geodesy, Cartography and Cadastre Authority of the Slovak Republic. This new conformal projection uses a normal cone, secant to the reference ellipsoid and was designed for the Slovak Republic territory by Margita Vajsáblová. The results are demonstrated in Tables 1 to 3. The valuation of scale distortions and their quadratic values in the mentioned cartographic projections by additive criterion has been calculated by Equation (8) for $n = 5\,424$ points (Table 2) and by integral criterion by Equation (7), where region Δ is limited by boundaries of the Slovak Republic (Table 3).

Conclusions

Disadvantages of currently used Křovák's projection such as large scale distortion became evident after dissolution of Czechoslovakia. The maximal scale distortion of this projection is about 4.3 cm/km larger than scale distortion

of Lambert's projection with parameters for the Slovak Republic also about 3.8 cm/km larger than scale distortion of oblique variational projection. Valuation of these map projections by variational criteria (additive or integral) demonstrated in the Tables 2 and 3 also confirms their suitability.

The most suitable cartographic projection of territory of the Slovak Republic by extremal criteria is the Lambert's projection with parameters for the Slovak Republic. The most suitable map projection by variational criteria is the above described variational projection in oblique position.

Conflict of Interests

The author(s) have not declared any conflict of interests.

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REFERENCES

- Airy GB (1861). Explanation of a projection by balance of errors for maps applying to a very large extent of the Earth's surface and comparison with other projections. *Philosophical Magaz. J. Sci.* 22:409-421.
- Bořík M (1999). Using of variational criteria for projection of CR. Map – product of geoinformatic technologies: Proceedings of the 13th Cartographic Conference. Bratislava. pp. 37-42.
- Chebyshev PL (1962). *Sur la construction des cartes géographiques. Oeuvres de P.L. Chebyshev.* Chelsea.
- Frankich K (1982). Optimization of geographic map projections for Canadian territory, Simon Fraser University, Canada.
- Hojovec V, Klášterka J, Rendlová H (1975). Using of Chebyshev's variational criterion in conformal projection. *Geodetic cartographic Rev.* 21(63):3-6.
- Hojovec V, Daniš M, Hájek M, Veverka B (1987). *Cartography.* Praha (Geodetic and Cartographic Enterprise).
- Kavraiskii VV (1959). *Collected works 2.* Moscow.
- Meshcheryakov GA (1968). *Theoretical foundations of mathematical cartography.* Moscow.
- Srnka E (1986). *Mathematical cartography.* Military Academy of Antonín Zápotocký, Brno.
- Urmaev NA (1947). *Methods for finding new cartographic projections.* Moscow.
- Vajsáblová M (2011). Proposal of new cartographic projection of Slovak Republic. *Geodetic and cartographic Rev.* 57(99):185-190.