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Exact solutions for the nonlinear KPP equation by using the Riccati equation method combined with the (G'/G)- expansion method

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The improved Riccati equation method combined with the improved (G'/G)- expansion method is an interesting approach to find more general exact solutions of the nonlinear evolution equations in mathematical physics. The objective of this article is to employ this method to construct exact solutions involving parameters of a nonlinear Kolmogorov-Petrovskii-Piskunov (KPP) equation. When these parameters are taken to be special values, the solitary wave solutions, the periodic wave solutions and the rational function solutions are derived from the exact solutions. The proposed method appears to be effective for solving other nonlinear evolution equations in the mathematical physics.

Key words: The Riccati equation method, the (G'/G)- expansion method, the nonlinear Kolmogorov-Petrovskii-Piskunov (KPP) equation, exact solutions, solitary wave solutions, periodic wave solutions, rational solutions.

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INTRODUCTION

Many problems in the branches of modern physics are described in terms of suitable nonlinear models, and nonlinear physical phenomena are related to nonlinear differential equations, which are involved in many fields from physics to biology, chemistry, mechanics, and so on. Nonlinear wave phenomena are very important in nonlinear science, in recent years, much effort has been spent on the construction of exact solutions of nonlinear partial differential solutions. Many effective methods to construct the exact solutions of these equations have been established, such as, the inverse scattering transform method (Ablowitz and Clarkson, 1991), the Hirota method (Hirota, 1971), the truncated expansion method(Weiss et al., 1983), the Backlund transform method (Miura, 1979; Rogers and Shadwick, 1982), the exp-function method (He and Wu, 2006; Yusufoglu, 2008), the tanh- function method (Fan, 2000; Zhang and Xia, 2008), the Jacobi elliptic function method (Chen and Wang, 2005; Lu, 2005), the (G'/G)–expansion method (Wang and Zhang, 2008; Feng and Wan, 2011; Zayed

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and Al-Joudi 2009; Zayed and Abdelaziz, 2010; Zayed and El-Malky, 2011), the modified simple equation (Jawad,et al .,2010; Zayed, 2011; Zayed and Hoda Ibrahim, 2012, Zayed and Hoda Ibrahim, 2014; Zayed and Arnous, 2012), the Riccati equation method (Zhu, 2008; Li and Zhang, 2010; Zayed and Arnous, 2013), the improved Riccati equation method (Li, 2012), the method of averaging (Leilei et al., 2014) and so on. The objective of this paper is to apply the improved Riccati equation method combined with the improved (G'/G) - expansion method to find the exact solutions of the following nonlinear Kolmogorov-Petrovskii-Piskunov (KPP) equation:

$$u_t - u_{xx} + \mu u + \gamma u^2 + \delta u^3 = 0, \qquad (1)$$

Where μ, γ, δ are real constants. Equation (1) is important in the physical fields, and includes the Fisher equation, the Huxley equation, the Burgers- Huxley equation, the Chaffee-Infanfe equation and the Fitzhugh-Nagumo equation. Equation (1) has been investigated recently in (Feng and Wan, 2011) using the (G'/G)-expansion method and in (Zayed and Hoda Ibrahim, 2014) using the modified simple equation method. The rest of this paper is organized as follows: First is a description of the improved Riccati equation method combined with the improved (G'/G)- expansion method. Next is application of this method to solve the nonlinear KPP equation (1). Thereafter, the physical explanations of the obtained results are given, and conclusions are obtained.

Description of the Riccati equation method combined with the (G'/G) - expansion method

Suppose that we have the following nonlinear evolution equation:

$$F(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0, \qquad (2)$$

Where F is a polynomial in $u(x, t)$ and its partial derivatives, in which the highest order derivatives and the nonlinear terms, are involved. In the following, we give the main steps of the Riccati equation method combined with the (G'/G) - expansion method (Li, 2012):

Step 1. We use the traveling wave transformation

$$u(x,t) = u(\xi), \quad \xi = kx + \omega t, \qquad (3)$$

Where k, ω are constants, to reduce Equation (1) to the following ordinary differential equation (ODE):

$$P(u, u', u'', \ldots) = 0, \qquad (4)$$

Where P is a polynomial in $u(\xi)$ and its total derivatives, while the dashes denote the derivatives with respect to ξ .

Step 2. We assumes that Equation (4) has the formal solution:

$$u(\xi) = \sum_{i=-n}^{n} \alpha_i [f(\xi)]^i, \qquad (5)$$

Where α_i $(i = -n, \ldots, n)$ are constants to be determined later $\alpha_n \neq 0$ *or* $\alpha_{-n} \neq 0$, while $f(\xi)$ satisfies the generalized Riccati equation:

$$f'(\xi) = p + rf(\xi) + qf^2(\xi), \qquad (6)$$

Where *p, r* and *q* are real constants, such that $q \neq 0$ and $f(\xi)$ will be determined in the Step 4 below.

Step 3. The positive integer *n* in Equation (5) can be determined by balancing the highest-order derivatives with the nonlinear terms appearing in Equation (4).

Step 4. We determine the solutions $f(\xi)$ of Equation (6) using the improved (G'/G)-expansion method, by assuming that its formal solution has the form

$$f(\xi) = \sum_{i=-m}^{m} \beta_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i, \qquad (7)$$

Where β_i $(i = -m, \ldots, m)$ are constants to be determined later $\beta_m \neq 0$ *or* $\beta_{-m} \neq 0$, and $G(\xi)$ satisfies the following linear ODE:

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \qquad (8)$$

Where λ and μ are constants.

Step 5. The positive integer *m* in Equation (7) can be determined by balancing $f'(\xi)$ and $f^2(\xi)$ in Equation (6) to get $m = 1$. Thus, the solution (7) reduces to.

$$f(\xi) = \beta_0 + \beta_1 \left[\frac{G'(\xi)}{G(\xi)}\right] + \beta_{-1} \left[\frac{G'(\xi)}{G(\xi)}\right]^{-1}, \qquad (9)$$

Where $\beta_0, \beta_1, \beta_{-1}$ are constants to be determined,such that $\beta_1 \neq 0$ *or* $\beta_{-1} \neq 0$. Substituting Equation (9) along with Equation (8) into Equation (6) and equating all the coefficients of powers of $\left(\frac{G'}{G}\right)$ to zero, yields a set of algebraic equations, which can be solved to get the following two cases:

Case 1

$$\beta_0 = \frac{-(\lambda + r)}{2q},\ \beta_1 = \frac{-1}{q},\ \beta_{-1} = 0,\ p = \frac{r^2 - \lambda^2 + 4\mu}{4q},\ q \neq 0.$$

In this case, the solution of Equation (6) has the form

$$f(\xi) = \frac{-(\lambda + r)}{2q} - \frac{1}{q}\left[\frac{G'(\xi)}{G(\xi)}\right]. \qquad (10)$$

Case 2

$$\beta_0 = \frac{\lambda - r}{2q},\ \beta_1 = 0,\ \beta_{-1} = \frac{\mu}{q},\ p = \frac{r^2 - \lambda^2 + 4\mu}{4q},\ q \neq 0.$$

In this case, the solution of Equation (6) has the form

$$f(\xi) = \frac{\lambda - r}{2q} + \frac{\mu}{q}\left[\frac{G'(\xi)}{G(\xi)}\right]^{-1}. \qquad (11)$$

From the Cases 1 and 2, we deduce that $\lambda^2 - 4\mu = r^2 - 4pq$. On solving Equation (8) we deduce that (G'/G) has the forms:

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} -\frac{\lambda}{2} + \frac{\sqrt{r^2-4pq}}{2}\left[\frac{c_1\sinh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right) + c_2\cosh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)}{c_1\cosh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right) + c_2\sinh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)}\right] & if\ \ r^2-4pq>0 \qquad (12) \\ -\frac{\lambda}{2} + \frac{\sqrt{4pq-r^2}}{2}\left[\frac{-c_1\sin\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right) + c_2\cos\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right)}{c_1\cos\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right) + c_2\sin\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right)}\right] & if\ \ r^2-4pq<0 \qquad (13) \\ -\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2\xi} & if\ \ r^2-4pq=0 \qquad (14) \end{cases}$$

Where c_1 and c_2 are arbitrary constants.

Step 6. Substituting Equation (5) along Equation (6) into Equation (4) and equating the coefficients of all powers of $f(\xi)$ to zero, we obtain a system of algebraic equations, which can be solved using the Maple or Mathematica to get the values of α_i, k and ω.\

Step 7. Substituting the values of α_i, k and ω as well as the solutions $f(\xi)$ given by Equation (10) or Equation (11) into Equation (5), we finally obtain the exact solutions of Equation (2) for the both Cases 1 and 2.

An application

Here we apply the proposed method just described to construct the exact solutions of the nonlinear KPP Equation (1). To the end, we use the wave transformation (3) to reduce Equation (1) to the following ODE:

$$\omega u'(\xi) - k^2 u''(\xi) + \mu u(\xi) + \gamma u^2(\xi) + \delta u^3(\xi) = 0. \qquad (15)$$

By balancing u'' *with* u^3, we have $n = 1$. Consequently, we have the formal solution

$$u(\xi) = \alpha_0 + \alpha_1 f(\xi) + \alpha_{-1} f^{-1}(\xi), \qquad (16)$$

Where $\alpha_0, \alpha_1, \alpha_{-1}$ are parameters to be determined later, such that $\alpha_{-1} \neq 0$ or $\alpha_1 \neq 0$.
Substituting Equation (16) along with Equation (6) into Equation (15) and equating the coefficients of all powers of $f(\xi)$ to zero, we get the following system of algebraic equations:

$$\begin{aligned}
f^3: &\quad -2k^2\alpha_1 q^2 + \delta\alpha_1^3 = 0,\\
f^2: &\quad \alpha_1 q\omega - 3\alpha_1 rqk^2 + \gamma\alpha_1^2 + 3\delta\alpha_0\alpha_1^2 = 0,\\
f\ : &\quad \omega\alpha_1 r - k^2(\alpha_1 r^2 + 2\alpha_1 pq) + \mu\alpha_1 + 2\gamma\alpha_0\alpha_1 + \delta(3\alpha_0^2\alpha_1 + 3\alpha_1^2\alpha_{-1}) = 0,\\
f^0: &\quad \omega(\alpha_1 p - \alpha_{-1} q) - k^2(\alpha_1 rp + r\alpha_{-1} q) + \mu\alpha_0 + \gamma(\alpha_0^2 + 2\alpha_1\alpha_{-1}) + \delta(\alpha_0^3 + 6\alpha_0\alpha_1\alpha_{-1}) = 0,\\
f^{-3}: &\quad -2k^2\alpha_{-1}p^2 + \delta\alpha_{-1}^3 = 0,\\
f^{-2}: &\quad -\alpha_{-1}p\omega - 3\alpha_{-1}rpk^2 + \gamma\alpha_{-1}^2 + 3\delta\alpha_0\alpha_{-1}^2 = 0,\\
f^{-1}: &\quad -\omega\alpha_{-1} r - k^2(\alpha_{-1} r^2 + 2\alpha_{-1} pq) + \mu\alpha_{-1} + 2\gamma\alpha_0\alpha_{-1} + \delta(3\alpha_0^2\alpha_{-1} + 3\alpha_{-1}^2\alpha_1) = 0.
\end{aligned}$$

By solving the above algebraic equations with the aid of Maple or Mathematical, we have the following results:

Result 1

$$\omega = \frac{\mu - k^2(r^2 - 4pq)}{\sqrt{r^2 - 4pq}},$$

$$\gamma = \frac{\left(2k^2(r^2-4pq) + \mu\right)\left((r^2-4pq) + r\sqrt{r^2-4pq}\right)\left((r^2-2pq) - r\sqrt{r^2-4pq}\right)}{4pq(r^2-4pq)\alpha_0},$$

$$\alpha_1 = 0,\ \delta = \frac{k^2\left((r^2-2pq) - r\sqrt{r^2-4pq}\right)}{\alpha_0^2},\ \alpha_{-1} = \alpha_0\left(\frac{r + \sqrt{r^2-4pq}}{2q}\right).$$

provided that $r^2 - 4pq > 0$.

Now, the solution for the result 1 becomes

$$u(\xi)=\alpha_0+\alpha_0\left(\frac{r+\sqrt{r^2-4pq}}{2q}\right)f^{-1}(\xi), \tag{17}$$

Where

$$\xi=kx+\left(\frac{\mu-k^2(r^2-4pq)}{\sqrt{r^2-4pq}}\right)t. \tag{18}$$

Substituting Equation (10) into Equation (17) and using Equations (12) to (14) we have the hyperbolic wave solutions of Equation (1) as follows:

$$u(\xi)=\alpha_0-\alpha_0\left(r+\sqrt{r^2-4pq}\right)\left\{r+\sqrt{r^2-4pq}\left[\frac{c_1\sinh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)+c_2\cosh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)}{c_1\cosh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)+c_2\sinh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)}\right]\right\}^{-1}. \tag{19}$$

Substituting the formulas (8), (10), (12) and (14) obtained by Peng (2009) into Equation (19), we have respectively the following exact solutions for Equation (1):

(i) If $|c_1|>|c_2|$, then

$$u_1(\xi)=\alpha_0-\alpha_0\left(r+\sqrt{r^2-4pq}\right)\left\{r+\sqrt{r^2-4pq}\tanh\left(\tfrac{\xi}{2}\sqrt{r^2-4pq}+\operatorname{sgn}(c_1c_2)\psi_1\right)\right\}^{-1} \tag{20}$$

Where $\psi_1=\tanh^{-1}\left(\frac{|c_2|}{|c_1|}\right)$.

(ii) If $|c_2|>|c_1|\neq 0,$ then

$$u_2(\xi)=\alpha_0-\alpha_0\left(r+\sqrt{r^2-4pq}\right)\left\{r+\sqrt{r^2-4pq}\coth\left(\tfrac{\xi}{2}\sqrt{r^2-4pq}+\operatorname{sgn}(c_1c_2)\psi_2\right)\right\}^{-1} \tag{21}$$

Where $\psi_2=\coth^{-1}\left(\frac{|c_2|}{|c_1|}\right)$.

(iii) If $|c_2|>|c_1|=0,$ then

$$u_3(\xi)=\alpha_0-\alpha_0\left(r+\sqrt{r^2-4pq}\right)\left\{r+\sqrt{r^2-4pq}\coth\left(\tfrac{\xi}{2}\sqrt{r^2-4pq}\right)\right\}^{-1}, \tag{22}$$

(iv) If $|c_1|=|c_2|,$ then we have the trivial solution which is rejected.

Substituting Equation (11) into Equation (17) and using Equations (12) to (14) we have the hyperbolic wave solutions of Equation (1) as follows:

$$u(\xi)=\alpha_0+\alpha_0\left(r+\sqrt{r^2-4pq}\right)\left\{\lambda-r+4\mu\left[-\lambda+\sqrt{r^2-4pq}\left(\frac{c_1\sinh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)+c_2\cosh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)}{c_1\cosh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)+c_2\sinh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)}\right)\right]^{-1}\right\}^{-1}. \tag{23}$$

Substituting the formulas (8), (10), (12) and (14) obtained by Peng (2009) into Equation (23), we have respectively the following exact solutions for Equation (1):

(i) If $|c_1|>|c_2|$, then

$$u_4(\xi)=\alpha_0+\alpha_0\left(r+\sqrt{r^2-4pq}\right)\left\{\lambda-r+4\mu\left[-\lambda+\sqrt{r^2-4pq}\tanh\left(\tfrac{\xi}{2}\sqrt{r^2-4pq}+\operatorname{sgn}(c_1c_2)\psi_1\right)\right]^{-1}\right\}^{-1} \tag{24}$$

Where $\psi_1=\tanh^{-1}\left(\frac{|c_2|}{|c_1|}\right)$.

(ii) If $|c_2|>|c_1|\neq 0,$ then

$$u_5(\xi)=\alpha_0+\alpha_0\left(r+\sqrt{r^2-4pq}\right)\left\{\lambda-r+4\mu\left[-\lambda+\sqrt{r^2-4pq}\coth\left(\tfrac{\xi}{2}\sqrt{r^2-4pq}+\operatorname{sgn}(c_1c_2)\psi_2\right)\right]^{-1}\right\}^{-1} \tag{25}$$

Where $\psi_2=\coth^{-1}\left(\frac{|c_2|}{|c_1|}\right)$.

(iii) If $|c_2|>|c_1|=0,$, then

$$u_6(\xi)=\alpha_0+\alpha_0\left(r+\sqrt{r^2-4pq}\right)\left\{\lambda-r+4\mu\left[-\lambda+\sqrt{r^2-4pq}\coth\left(\tfrac{\xi}{2}\sqrt{r^2-4pq}\right)\right]^{-1}\right\}^{-1}, \tag{26}$$

(iv) If $|c_2|=|c_1|,$ then

$$u_7(\xi)=\alpha_0+\alpha_0\left(r+\sqrt{r^2-4pq}\right)\left\{\lambda-r+4\mu\left[-\lambda+\sqrt{r^2-4pq}\right]^{-1}\right\}^{-1}. \tag{27}$$

Result 2. Consider

$$\omega=\frac{k^2(r^2-4pq)-\mu}{\sqrt{r^2-4pq}},$$

$$\gamma=\frac{\left(2k^2(r^2-4pq)+\mu\right)\left((r^2-4pq)+r\sqrt{r^2-4pq}\right)\left((r^2-2pq)-r\sqrt{r^2-4pq}\right)}{4pq(r^2-4pq)\alpha_0},$$

$$\alpha_{-1}=0,\ \delta=\frac{k^2\left((r^2-2pq)-r\sqrt{r^2-4pq}\right)}{\alpha_0^2},\ \alpha_1=\alpha_0\left(\frac{r+\sqrt{r^2-4pq}}{2q}\right).$$

Now, the solution for the result 2, becomes

$$u(\xi)=\alpha_0+\alpha_0\left(\frac{r+\sqrt{r^2-4pq}}{2q}\right)f(\xi), \tag{28}$$

Where $\xi=kx+\left(\frac{k^2(r^2-4pq)-\mu}{\sqrt{r^2-4pq}}\right)t.$

Substituting Equation (10) into Equation (28) and using Equations (12) to (14) we have the hyperbolic wave solutions of Equation (1) as follows:

$$u(\xi)=\alpha_0-\frac{\alpha_0}{4q^2}\left(r+\sqrt{r^2-4pq}\right)\left\{r+\sqrt{r^2-4pq}\left[\frac{c_1\sinh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)+c_2\cosh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)}{c_1\cosh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)+c_2\sinh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)}\right]\right\} \quad (29)$$

Substituting the formulas (8), (10), (12) and (14) obtained by Peng (2009) into Equation (29), we have respectively the following exact solutions for Equation (1):

(i) If $|c_1|>|c_2|$, then

$$u_8(\xi)=\alpha_0-\frac{\alpha_0}{4q^2}\left(r+\sqrt{r^2-4pq}\right)\left\{r+\sqrt{r^2-4pq}\tanh\left(\frac{\xi}{2}\sqrt{r^2-4pq}+\operatorname{sgn}(c_1c_2)\psi_1\right)\right\}, \quad (30)$$

Where $\psi_1=\tanh^{-1}\left(\frac{|c_2|}{|c_1|}\right)$.

(ii) If $|c_2|>|c_1|\neq 0$, then

$$u_9(\xi)=\alpha_0-\frac{\alpha_0}{4q^2}\left(r+\sqrt{r^2-4pq}\right)\left\{r+\sqrt{r^2-4pq}\coth\left(\frac{\xi}{2}\sqrt{r^2-4pq}+\operatorname{sgn}(c_1c_2)\psi_2\right)\right\}, \quad (31)$$

Where $\psi_2=\coth^{-1}\left(\frac{|c_2|}{|c_1|}\right)$.

(iii) If $|c_2|>|c_1|=0$, then

$$u_{10}(\xi)=\alpha_0-\frac{\alpha_0}{4q^2}\left(r+\sqrt{r^2-4pq}\right)\left\{r+\sqrt{r^2-4pq}\coth\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)\right\}, \quad (32)$$

(iv) If $|c_2|=|c_1|$, then

$$u_{11}(\xi)=\alpha_0-\frac{\alpha_0}{4q^2}\left(r+\sqrt{r^2-4pq}\right)^2, \quad (33)$$

Substituting Equation (11) into Equation (28) and using Equation (12) to (14) we have the hyperbolic wave solutions of Equation (1) as follows:

$$u(\xi)=\alpha_0+\frac{\alpha_0}{4q^2}\left(r+\sqrt{r^2-4pq}\right)\left\{\lambda-r+4\mu\left[-\lambda+\sqrt{r^2-4pq}\left(\frac{c_1\sinh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)+c_2\cosh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)}{c_1\cosh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)+c_2\sinh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)}\right)\right]^{-1}\right\} \quad (34)$$

Substituting the formulas (8), (10), (12) and (14) obtained by Peng (2009) into Equation (34), we have respectively the following exact solutions for Equation (1):

(i) If $|c_1|>|c_2|$, then

$$u_{12}(\xi)=\alpha_0+\frac{\alpha_0}{4q^2}\left(r+\sqrt{r^2-4pq}\right)\left\{\lambda-r+4\mu\left[-\lambda+\sqrt{r^2-4pq}\tanh\left(\frac{\xi}{2}\sqrt{r^2-4pq}+\operatorname{sgn}(c_1c_2)\psi_1\right)\right]^{-1}\right\} \quad (35)$$

Where $\psi_1=\tanh^{-1}\left(\frac{|c_2|}{|c_1|}\right)$.

(ii) If $|c_2|>|c_1|\neq 0$, then

$$u_{13}(\xi)=\alpha_0+\frac{\alpha_0}{4q^2}\left(r+\sqrt{r^2-4pq}\right)\left\{\lambda-r+4\mu\left[-\lambda+\sqrt{r^2-4pq}\coth\left(\frac{\xi}{2}\sqrt{r^2-4pq}+\operatorname{sgn}(c_1c_2)\psi_2\right)\right]^{-1}\right\} \quad (36)$$

Where $\psi_2=\coth^{-1}\left(\frac{|c_2|}{|c_1|}\right)$.

(iii) If $|c_2|>|c_1|=0$, then

$$u_{14}(\xi)=\alpha_0+\frac{\alpha_0}{4q^2}\left(r+\sqrt{r^2-4pq}\right)\left\{\lambda-r+4\mu\left[-\lambda+\sqrt{r^2-4pq}\coth\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)\right]^{-1}\right\} \quad (37)$$

(iv) If $|c_2|=|c_1|$, then

$$u_{15}(\xi)=\alpha_0+\frac{\alpha_0}{4q^2}\left(r+\sqrt{r^2-4pq}\right)\left\{\lambda-r+4\mu\left[-\lambda+\sqrt{r^2-4pq}\right]^{-1}\right\} \quad (38)$$

Result 3. Consider

$$\omega=\pm k\sqrt{k^2(r^2-4pq)+2\mu},$$

$$\gamma=\frac{\left(k^2(r^2-2pq)+\mu+rk\sqrt{k^2(r^2-4pq)+2\mu)}\right)\left(k^2(r^2-4pq)+2\mu-rk\sqrt{k^2(r^2-4pq)+2\mu)}\right)}{p\alpha_0^2\left(2k^2pq-\mu\right)},$$

$$\alpha_1=0,\delta=\frac{k^2(r^2-2pq)+\mu\pm rk\sqrt{k^2(r^2-4pq)+2\mu}}{\alpha_0^2},$$

$$\alpha_{-1}=\frac{kp\alpha_0\left(kr\mp\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}.$$

Now, the solution for the result 3 becomes

$$u(\xi)=\alpha_0+\frac{kp\alpha_0\left(kr\mp\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}f^{-1}(\xi) \quad (39)$$

Where

$$\xi=kx\pm k\sqrt{k^2(r^2-4pq)+2\mu}\,t, \quad \text{and}$$
$$k^2(r^2-4pq)+2\mu\geq 0. \quad (40)$$

Substituting Equation (10) into Equation (39) and using Equations (12) to (14) we have the exact solutions of Equation (1) as follows:

If $r^2-4pq>0$, we have the hyperbolic wave solutions

$$u(\xi)=\alpha_0-\frac{kp\alpha_0\left(kr\mp\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{r}{2q}+\frac{\sqrt{r^2-4pq}}{2q}\left[\frac{c_1\sinh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)+c_2\cosh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)}{c_1\cosh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)+c_2\sinh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)}\right]\right\}^{-1} \quad (41)$$

Substituting the formulas (8), (10), (12) and (14) obtained by Peng (2009) into Equation (41), we have respectively the following exact solutions for Equation (1):

(i) If $|c_1| > |c_2|$, then

$$u_{16}(\xi) = \alpha_0 - \frac{kp\alpha_0\left(kr \mp \sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{r}{2q}+\frac{\sqrt{r^2-4pq}}{2q}\tanh\left(\tfrac{\xi}{2}\sqrt{r^2-4pq}+\operatorname{sgn}(c_1c_2)\psi_1\right)\right\}^{-1} \quad (42)$$

Where $\psi_1 = \tanh^{-1}\left(\dfrac{|c_2|}{|c_1|}\right)$.

(ii) If $|c_2| > |c_1| \neq 0$, then

$$u_{17}(\xi) = \alpha_0 - \frac{kp\alpha_0\left(kr \mp \sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{r}{2q}+\frac{\sqrt{r^2-4pq}}{2q}\coth\left(\tfrac{\xi}{2}\sqrt{r^2-4pq}+\operatorname{sgn}(c_1c_2)\psi_2\right)\right\}^{-1} \quad (43)$$

Where $\psi_2 = \coth^{-1}\left(\dfrac{|c_2|}{|c_1|}\right)$.

(iii) If $|c_2| > |c_1| = 0$, then

$$u_{18}(\xi) = \alpha_0 - \frac{kp\alpha_0\left(kr \mp \sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{r}{2q}+\frac{\sqrt{r^2-4pq}}{2q}\coth\left(\tfrac{\xi}{2}\sqrt{r^2-4pq}\right)\right\}^{-1}. \quad (44)$$

(iv) If $|c_2| = |c_1|$, then

$$u_{19}(\xi) = \alpha_0 - \frac{kp\alpha_0\left(kr \mp \sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{r}{2q}+\frac{\sqrt{r^2-4pq}}{2q}\right\}^{-1}. \quad (45)$$

If $r^2-4pq<0$, we have the trigonometric wave solutions

$$u(\xi) = \alpha_0 - \frac{kp\alpha_0\left(kr \mp \sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{r}{2q}+\frac{\sqrt{4pq-r^2}}{2q}\left[\frac{-c_1\sin\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right)+c_2\cos\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right)}{c_1\cos\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right)+c_2\sin\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right)}\right]\right\}^{-1} \quad (46)$$

Now, we can simplify Equation (46) to get the following periodic wave solutions:

$$u_{20}(\xi) = \alpha_0 - \frac{kp\alpha_0\left(kr \mp \sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{r}{2q}+\frac{\sqrt{4pq-r^2}}{2q}\tan\left(\xi_1-\tfrac{\xi}{2}\sqrt{4pq-r^2}\right)\right\}^{-1}, \quad (47)$$

Where $\xi_1 = \tan^{-1}\left(\dfrac{c_2}{c_1}\right)$,

and

$$u_{21}(\xi) = \alpha_0 - \frac{kp\alpha_0\left(kr \mp \sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{r}{2q}+\frac{\sqrt{4pq-r^2}}{2q}\cot\left(\xi_2+\tfrac{\xi}{2}\sqrt{4pq-r^2}\right)\right\}^{-1} \quad (48)$$

Where $\xi_2 = \cot^{-1}\left(\dfrac{c_2}{c_1}\right)$.

If $r^2-4pq=0$, we have the rational wave solutions

$$u_{22}(\xi) = \alpha_0 - \frac{kp\alpha_0\left(kr \mp \sqrt{2\mu}\right)}{2k^2pq-\mu}\left\{\frac{r}{2q}+\frac{1}{q}\left(\frac{c_2}{c_1+c_2\xi}\right)\right\}^{-1}, \quad (49)$$

Where c_1, c_2 are arbitrary constants.

Substituting Equation (11) into Equation (39) and using Equations (12) to (14) we have the exact solutions of Equation (1) as follows:

If $r^2-4pq>0$, we have the hyperbolic wave solutions

$$u(\xi) = \alpha_0 + \frac{kp\alpha_0\left(kr \mp \sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{\lambda-r}{2q}+\frac{\mu}{q}\left[-\frac{\lambda}{2}+\frac{\sqrt{r^2-4pq}}{2}\left(\frac{c_1\sinh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)+c_2\cosh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)}{c_1\cosh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)+c_2\sinh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)}\right)\right]^{-1}\right\}^{-1} \quad (50)$$

Substituting the formulas (8), (10), (12) and (14) obtained by Peng (2009) into Equation (50), we have respectively the following exact solutions for Equation (1):

(i) If $|c_1| > |c_2|$, then

$$u_{23}(\xi) = \alpha_0 + \frac{kp\alpha_0\left(kr \mp \sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{\lambda-r}{2q}+\frac{\mu}{q}\left[-\frac{\lambda}{2}+\frac{\sqrt{r^2-4pq}}{2}\tanh\left(\tfrac{\xi}{2}\sqrt{r^2-4pq}+\operatorname{sgn}(c_1c_2)\psi_1\right)\right]^{-1}\right\}^{-1} \quad (51)$$

Where $\psi_1 = \tanh^{-1}\left(\dfrac{|c_2|}{|c_1|}\right)$.

(ii) If $|c_2| > |c_1| \neq 0$, then

$$u_{24}(\xi) = \alpha_0 + \frac{kp\alpha_0\left(kr \mp \sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{\lambda-r}{2q}+\frac{\mu}{q}\left[-\frac{\lambda}{2}+\frac{\sqrt{r^2-4pq}}{2}\coth\left(\tfrac{\xi}{2}\sqrt{r^2-4pq}+\operatorname{sgn}(c_1c_2)\psi_2\right)\right]^{-1}\right\}^{-1}, \quad (52)$$

Where $\psi_2 = \coth^{-1}\left(\dfrac{|c_2|}{|c_1|}\right)$.

(iii) If $|c_2| > |c_1| = 0$, then

$$u_{25}(\xi) = \alpha_0 + \frac{kp\alpha_0\left(kr \mp \sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{\lambda-r}{2q}+\frac{\mu}{q}\left[-\frac{\lambda}{2}+\frac{\sqrt{r^2-4pq}}{2}\coth\left(\tfrac{\xi}{2}\sqrt{r^2-4pq}\right)\right]^{-1}\right\}^{-1} \quad (53)$$

(iv) If $|c_2| = |c_1|$, then

$$u_{26}(\xi) = \alpha_0 + \frac{kp\alpha_0\left(kr \mp \sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{\lambda-r}{2q}+\frac{\mu}{q}\left[-\frac{\lambda}{2}+\frac{\sqrt{r^2-4pq}}{2}\right]^{-1}\right\}^{-1} \quad (54)$$

If $r^2-4pq<0$, we have the trigonometric wave solutions

$$u(\xi)=\alpha_0+\frac{kp\alpha_0\left(kr\mp\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{\lambda-r}{2q}+\frac{\mu}{q}\left[-\frac{\lambda}{2}+\frac{\sqrt{4pq-r^2}}{2}\left(\frac{-c_1\sin\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right)+c_2\cos\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right)}{c_1\cos\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right)+c_2\sin\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right)}\right)\right]^{-1}\right\}^{-1} \quad (55)$$

Now, we can simplify Equation (55) to get the following periodic wave solutions:

$$u_{27}(\xi)=\alpha_0+\frac{kp\alpha_0\left(kr\mp\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{\lambda-r}{2q}+\frac{\mu}{q}\left[-\frac{\lambda}{2}+\frac{\sqrt{4pq-r^2}}{2}\tan\left(\xi_1-\frac{\xi}{2}\sqrt{4pq-r^2}\right)\right]^{-1}\right\}^{-1} \quad (56)$$

Where $\xi_1=\tan^{-1}\left(\dfrac{c_2}{c_1}\right)$,

and

$$u_{28}(\xi)=\alpha_0+\frac{kp\alpha_0\left(kr\mp\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{\lambda-r}{2q}+\frac{\mu}{q}\left[-\frac{\lambda}{2}+\frac{\sqrt{4pq-r^2}}{2}\cot\left(\xi_2+\frac{\xi}{2}\sqrt{4pq-r^2}\right)\right]^{-1}\right\}^{-1}, \quad (57)$$

Where $\xi_2=\cot^{-1}\left(\dfrac{c_2}{c_1}\right)$.

If $r^2-4pq=0$, we have the rational wave solutions

$$u_{29}(\xi)=\alpha_0+\frac{kp\alpha_0\left(kr\mp\sqrt{2\mu}\right)}{2k^2pq-\mu}\left\{\frac{\lambda-r}{2q}+\frac{\mu}{q}\left(\frac{-\lambda}{2}+\frac{c_2}{c_1+c_2\xi}\right)^{-1}\right\}^{-1}, \quad (58)$$

Where c_1, c_2 are arbitrary constants.

Result 4. Consider

$$\omega=\pm k\sqrt{k^2(r^2-4pq)+2\mu},$$
$$\gamma=\frac{\left(k^2(r^2-2pq)+\mu+rk\sqrt{k^2(r^2-4pq)+2\mu}\right)\left(k^2(r^2-4pq)+2\mu-rk\sqrt{k^2(r^2-4pq)+2\mu}\right)}{q\alpha_0^2\left(2k^2pq-\mu\right)},$$
$$\alpha_{-1}=0,\delta=\frac{k^2(r^2-2pq)+\mu\pm rk\sqrt{k^2(r^2-4pq)+2\mu}}{\alpha_0^2},\alpha_1=\frac{kq\alpha_0\left(kr\pm\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}.$$

Now, the solution for the result 4 becomes

$$u(\xi)=\alpha_0+\frac{kq\alpha_0\left(kr\pm\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}f(\xi) \quad (59)$$

Where $\xi=kx\pm k\sqrt{k^2(r^2-4pq)+2\mu}\,t$, and

$$k^2(r^2-4pq)+2\mu\geq 0. \quad (60)$$

Substituting Equation (10) into Equation (59) and using Equations (12) to(14) we have the exact solutions of Equation (1) as follows:

If $r^2-4pq>0$, we have the hyperbolic wave solutions

$$u(\xi)=\alpha_0-\frac{kq\alpha_0\left(kr\pm\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{r}{2q}+\frac{\sqrt{r^2-4pq}}{2q}\left[\frac{c_1\sinh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)+c_2\cosh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)}{c_1\cosh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)+c_2\sinh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)}\right]\right\}^{-1} \quad (61)$$

Substituting the formulas (8), (10), (12) and (14) obtained by Peng (2009) into Equation (61), we have respectively the following exact solutions for Equation (1):

(i) If $|c_1|>|c_2|$, then

$$u_{30}(\xi)=\alpha_0-\frac{kq\alpha_0\left(kr\pm\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{r}{2q}+\frac{\sqrt{r^2-4pq}}{2q}\tanh\left(\frac{\xi}{2}\sqrt{r^2-4pq}+\operatorname{sgn}(c_1c_2)\psi_1\right)\right\}^{-1} \quad (62)$$

Where $\psi_1=\tanh^{-1}\left(\dfrac{|c_2|}{|c_1|}\right)$.

(ii) If $|c_2|>|c_1|\neq 0$, then

$$u_{31}(\xi)=\alpha_0-\frac{kq\alpha_0\left(kr\pm\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{r}{2q}+\frac{\sqrt{r^2-4pq}}{2q}\coth\left(\frac{\xi}{2}\sqrt{r^2-4pq}+\operatorname{sgn}(c_1c_2)\psi_2\right)\right\}^{-1} \quad (63)$$

Where $\psi_2=\coth^{-1}\left(\dfrac{|c_2|}{|c_1|}\right)$.

(iii) If $|c_2|>|c_1|=0$, then

$$u_{32}(\xi)=\alpha_0-\frac{kq\alpha_0\left(kr\pm\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{r}{2q}+\frac{\sqrt{r^2-4pq}}{2q}\coth\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)\right\}^{-1}. \quad (64)$$

(iv) If $|c_2|=|c_1|$, then

$$u_{33}(\xi)=\alpha_0-\frac{kq\alpha_0\left(kr\pm\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{r}{2q}\pm\frac{\sqrt{r^2-4pq}}{2q}\right\}^{-1}. \quad (65)$$

If $r^2-4pq<0$, we have the trigonometric wave solutions

$$u(\xi)=\alpha_0-\frac{kq\alpha_0\left(kr\pm\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{r}{2q}+\frac{\sqrt{4pq-r^2}}{2q}\left[\frac{-c_1\sin\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right)+c_2\cos\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right)}{c_1\cos\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right)+c_2\sin\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right)}\right]\right\}^{-1} \quad (66)$$

Now, we can simplify Equation (66) to get the following periodic wave solution:

$$u_{34}(\xi)=\alpha_0-\frac{kq\alpha_0\left(kr\pm\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{r}{2q}+\frac{\sqrt{4pq-r^2}}{2q}\tan\left(\xi_1-\frac{\xi}{2}\sqrt{4pq-r^2}\right)\right\}^{-1}, \quad (67)$$

Where $\xi_1=\tan^{-1}\left(\dfrac{c_2}{c_1}\right)$,

$$u_{35}(\xi)=\alpha_0-\frac{kq\alpha_0\left(kr\pm\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{r}{2q}+\frac{\sqrt{4pq-r^2}}{2q}\cot\left(\xi_2+\tfrac{\xi}{2}\sqrt{4pq-r^2}\right)\right\}^{-1}, \quad (68)$$

Where $\xi_2=\cot^{-1}\left(\frac{c_2}{c_1}\right)$.

If $r^2-4pq=0$, we have the rational wave solutions

$$u_{36}(\xi)=\alpha_0-\frac{kq\alpha_0\left(kr\pm\sqrt{2\mu}\right)}{2k^2pq-\mu}\left\{\frac{r}{2q}+\frac{1}{q}\left(\frac{c_2}{c_1+c_2\xi}\right)\right\}^{-1}, \quad (69)$$

Where c_1, c_2 are arbitrary constants.

Substituting Equation (11) into Equation (59) and using Equations (12) to (14) we have the exact solutions of Equation (1) as follows:

If $r^2-4pq>0$, we have the hyperbolic wave solutions

$$u(\xi)=\alpha_0+\frac{kq\alpha_0\left(kr\pm\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{\lambda-r}{2q}+\frac{\mu}{q}\left[-\frac{\lambda}{2}+\frac{\sqrt{r^2-4pq}}{2}\,\frac{c_1\sinh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)+c_2\cosh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)}{c_1\cosh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)+c_2\sinh\left(\frac{\xi}{2}\sqrt{r^2-4pq}\right)}\right]^{-1}\right\}^{-1} \quad (70)$$

Substituting the formulas (8), (10), (12) and (14) obtained by Peng (2009) into Equation (70), we have respectively the following exact solutions for Equation (1):

(i) If $|c_1|>|c_2|$, then

$$u_{37}(\xi)=\alpha_0+\frac{kq\alpha_0\left(kr\pm\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{\lambda-r}{2q}+\frac{\mu}{q}\left[-\frac{\lambda}{2}+\frac{\sqrt{r^2-4pq}}{2}\tanh\left(\tfrac{\xi}{2}\sqrt{r^2-4pq}+\mathrm{sgn}(c_1c_2)\psi_1\right)\right]^{-1}\right\}^{-1}, \quad (71)$$

Where $\psi_1=\tanh^{-1}\left(\frac{|c_2|}{|c_1|}\right)$.

(ii) If $|c_2|>|c_1|\neq 0$, then

$$u_{38}(\xi)=\alpha_0+\left(\frac{k^2pr\alpha_0\pm\alpha_0k\,p\sqrt{k^2(r^2-4pq)+2\mu}}{2k^2pq-\mu}\right)\left\{\frac{\lambda-r}{2q}+\frac{\mu}{q}\left[-\frac{\lambda}{2}+\frac{\sqrt{r^2-4pq}}{2}\coth\left(\tfrac{\xi}{2}\sqrt{r^2-4pq}+\mathrm{sgn}(c_1c_2)\psi_2\right)\right]^{-1}\right\}^{-1}, \quad (72)$$

Where $\psi_2=\coth^{-1}\left(\frac{|c_2|}{|c_1|}\right)$.

(iii) $|c_2|>|c_1|=0$, then

$$u_{39}(\xi)=\alpha_0+\left(\frac{k^2pr\alpha_0\pm\alpha_0k\,p\sqrt{k^2(r^2-4pq)+2\mu}}{2k^2pq-\mu}\right)\left\{\frac{\lambda-r}{2q}+\frac{\mu}{q}\left[-\frac{\lambda}{2}+\frac{\sqrt{r^2-4pq}}{2}\coth\left(\tfrac{\xi}{2}\sqrt{r^2-4pq}\right)\right]^{-1}\right\}^{-1} \quad (73)$$

(iv) $|c_2|=|c_1|$, then

$$u_{40}(\xi)=\alpha_0+\left(\frac{k^2pr\alpha_0\pm\alpha_0k\,p\sqrt{k^2(r^2-4pq)+2\mu}}{2k^2pq-\mu}\right)\left\{\frac{\lambda-r}{2q}+\frac{\mu}{q}\left[-\frac{\lambda}{2}+\frac{\sqrt{r^2-4pq}}{2}\right]^{-1}\right\}^{-1}, \quad (74)$$

If $r^2-4pq<0$, we have the trigonometric wave solutions

$$u(\xi)=\alpha_0+\frac{kq\alpha_0\left(kr\pm\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{\lambda-r}{2q}+\frac{\mu}{q}\left[-\frac{\lambda}{2}+\frac{\sqrt{4pq-r^2}}{2}\left(\frac{-c_1\sin\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right)+c_2\cos\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right)}{c_1\cos\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right)+c_2\sin\left(\frac{\xi}{2}\sqrt{4pq-r^2}\right)}\right)\right]^{-1}\right\}^{-1} \quad (75)$$

Now, we can simplify Equation (75) to get the following periodic wave solution:

$$u_{41}(\xi)=\alpha_0+\frac{kq\alpha_0\left(kr\pm\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{\lambda-r}{2q}+\frac{\mu}{q}\left[-\frac{\lambda}{2}+\frac{\sqrt{4pq-r^2}}{2}\tan\left(\xi_1-\tfrac{\xi}{2}\sqrt{4pq-r^2}\right)\right]^{-1}\right\}^{-1}, \quad (76)$$

Where $\xi_1=\tan^{-1}\left(\frac{c_2}{c_1}\right)$,

$$u_{42}(\xi)=\alpha_0+\frac{kq\alpha_0\left(kr\pm\sqrt{k^2(r^2-4pq)+2\mu}\right)}{2k^2pq-\mu}\left\{\frac{\lambda-r}{2q}+\frac{\mu}{q}\left[-\frac{\lambda}{2}+\frac{\sqrt{4pq-r^2}}{2}\cot\left(\tfrac{\xi}{2}\sqrt{4pq-r^2}+\xi_2\right)\right]^{-1}\right\}^{-1}, \quad (77)$$

Where $\xi_2=\cot^{-1}\left(\frac{c_2}{c_1}\right)$.

If $r^2-4pq=0$, we have the rational wave solutions

$$u_{43}(\xi)=\alpha_0+\frac{kq\alpha_0\left(kr\pm\sqrt{2\mu}\right)}{2k^2pq-\mu}\left\{\frac{\lambda-r}{2q}+\frac{\mu}{q}\left(\frac{-\lambda}{2}+\frac{c_2}{c_1+c_2\xi}\right)^{-1}\right\}^{-1} \quad (78)$$

Where c_1, c_2 are arbitrary constants.

Physical explanations of our obtained solutions

Solitary, periodic and rational waves can be obtained from the exact solutions by setting particular values in its unknown parameters. Here, we have presented some graphs of solitary and periodic waves constructed by taking suitable values of involved unknown parameters to visualize the underlying mechanism of the original Equation (1). By using the mathematical software Maple, the plots of some obtained solutions have been shown in Figures 1 to 4. The obtained solutions of Equation (1) incorporate three types of explicit solutions, namely the hyperbolic, trigonometric and rational solutions.

Some conclusions

We have used the Riccati equation method combined

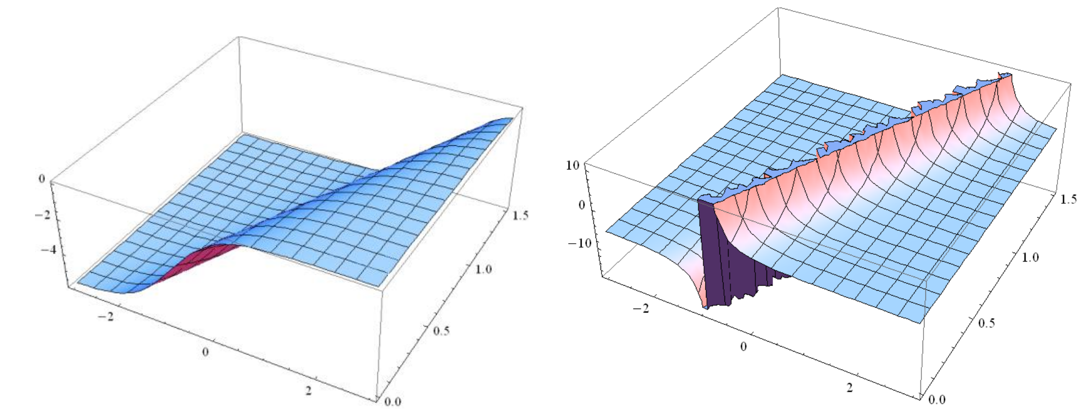

Figure 1. The plot of solutions u_1, u_2 with $\alpha_0 = p = q = k = \mu = 1, r = 3.$

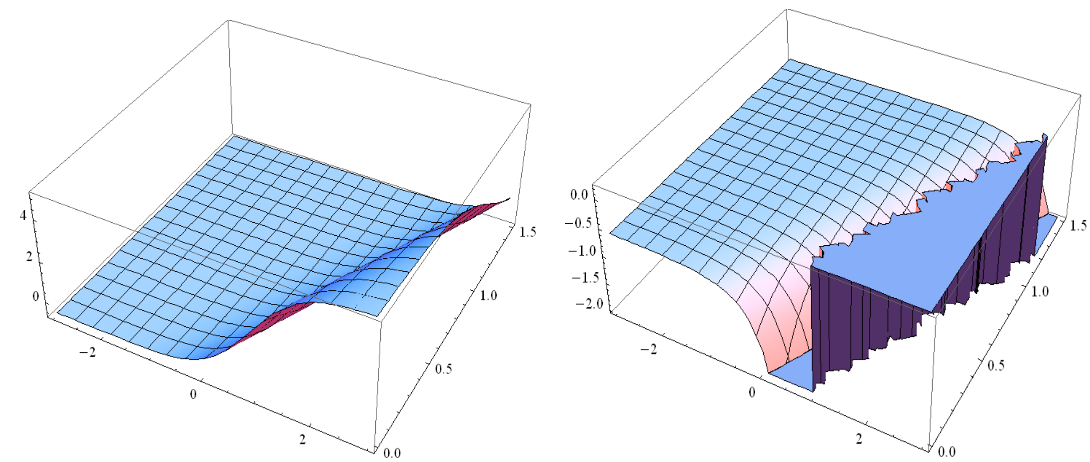

Figure 2. The plot of solutions u_4, u_5 with $\alpha_0 = p = q = k = \mu = \lambda = 1, r = 3.$

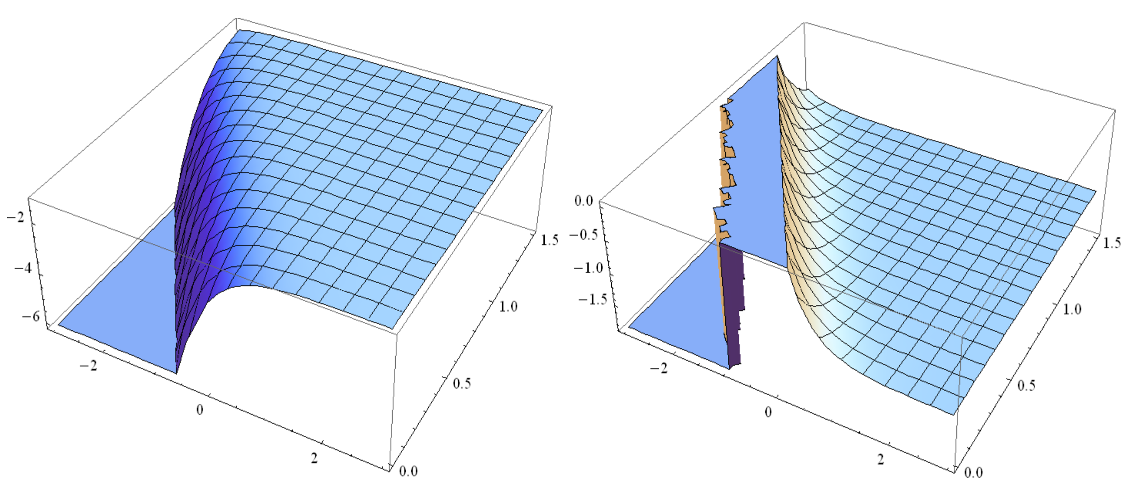

Figure 3. The plot of solutions u_{16}, u_{17} with $\alpha_0 = p = q = k = \mu = 1, r = 3.$

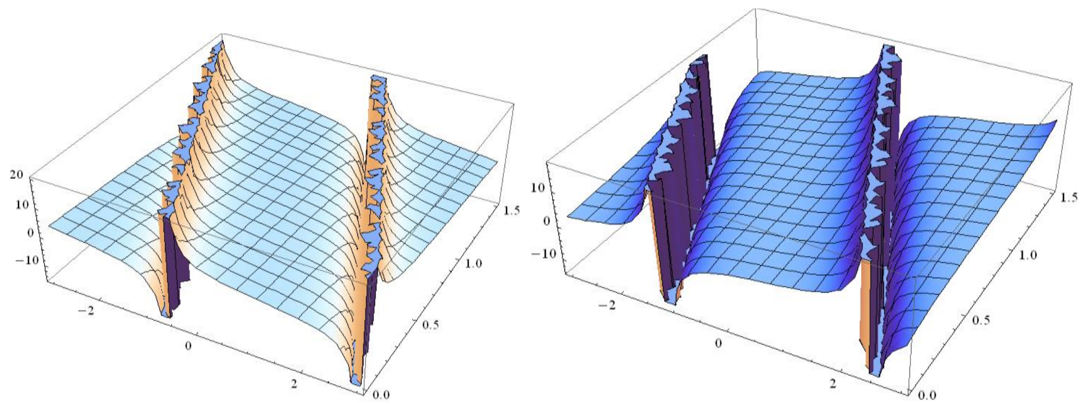

Figure 4. The plot of solutions u_{20}, u_{21} with $\alpha_0 = p = q = k = r = 1, \mu = 3.$

with the (G'/G)- expansion method to construct many new exact solutions of the nonlinear KPP Equation (1) involving parameters, which is expressed by the hyperbolic functions, the trigonometric functions and the rational functions. When the parameters are taken as special values the proposed method provides not only solitary wave solutions but also periodic wave solutions and rational wave solutions. These solutions will be of great importance for analyzing the nonlinear phenomena arising in applied physical sciences. This work shows that the proposed method is sufficient, effective and suitable for solving other nonlinear evolution equations in mathematical physics. Finally on comparing our results in this article with the results obtained in Feng et al. (2011) and Zayed and Hoda Ibrahim (2014), we conclude that our results are new and not reported elsewhere.

Conflict of Interest

The authors have not declared any conflict of interest.

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