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Full Length Research Paper

Post-buckling analysis of a simply supported beam under uniform thermal loading

Şeref Doğuşcan Akbaş* and Turgut Kocatürk

Yildiz Technical University, Department of Civil Engineering, Davutpaşa Campus, 34210 Esenler-İstanbul, Turkey.

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This paper focuses on post-buckling analysis of a simply supported beam subjected to a uniform thermal loading. The material of the beam was assumed as isotropic and hyper elastic. Both ends of the beam were supported by pins (pinned-pinned beam). In this study, finite element model of the beam was constructed by using total Lagrangian finite element model of two dimensional continuums for an eight-node quadratic element. The considered highly non-linear problem was solved by using incremental displacement-based finite element method in conjunction with Newton-Raphson iteration method. Based on the above mentioned solution procedure, analysis of large thermal bending and buckling/post buckling responses of the beam subjected transversally uniform temperature rise and with immovably pinned-pinned ends were presented. Characteristic curves showing the relationships between the beam displacements and temperature rise were illustrated. The results are compared with the published results obtained by using Timoshenko beam theory. Numerical results showed that the results of two dimensional continuum model and those of Timoshenko beam theory differ from each other with decrease of the slenderness of the beam. Therefore it is necessary to use a finite element model of two dimensional continuums in modelling the beam in the case of small slenderness.

Key words: Post-buckling analysis, total Lagrangian finite element model, two dimensional solid continuum, uniform temperature rise.

INTRODUCTION

In recent years, with the development of technology in aerospace engineering, robotics and manufacturing make it inevitable to excessively use non-linear models that must be solved numerically. Because, closed-form solutions of large-deflection problems of beams with general loading and boundary conditions using elliptic integrals are limited. Also, it is known that buckling problems are geometrically nonlinear problems. In the case of beams with immovable ends, temperature rise causes compressive forces end therefore buckling phenomena occurs. In recent years, much more attention has been given to the thermal buckling of beam structures. But most of the studies focused on the analysis of thermal buckling and post buckling of Euler-Bernoulli and Timoshenko beams. Rao and Raju (1984) investigated thermal post-buckling of columns. Global descriptions of the properties of buckled states of nonlinearly thermo-elastic beams and plates when heated at their ends and edges were investigated by Gauss and Antman (1984). Jekot (1996) investigated the thermal post-buckling of a beam made of physically nonlinear thermoelastic material by using the geometric equations in the von-Karman strain-displacement approximation.

Li (2000) examined thermal post-buckling of rods with pinned-fixed ends using the shooting method. Coffin and Bloom (1999) gave an elliptic integral solution for the symmetric post-buckling response of a linear elastic and hygrothermal beam with the two ends pinned. On the basis of exact nonlinear geometric theory of extensible beam and by using a shooting method, computational analysis for thermal post-buckling behaviour of beams with pinned-pinned, fixed-fixed and pinned-fixed ends were presented by Li and Cheng (2000), Li et al. (2002) and Li and Zhou (2001). Thermal post-buckling responses of an elastic beam, with immovably simply

*Corresponding author. E-mail: sakbas@yildiz.edu.tr. Tel: +90 212 383 51 51. Fax: +90 212 383 51 02.

supported ends and subjected to a transversely non-uniformly distributed temperature rising, were investigated by Li et al. (2003). Thermal post-buckling response of an immovably pinned-fixed Timoshenko beam subjected to a static transversely non uniform temperature rise is numerically analyzed by using a shooting method by Li and Zhou (2003).

Based on the finite element method, the analysis of heat conduction and structural stress and buckling were considered at the same time in the design optimization procedure by Chen et al. (2003). Vaz and Solano (2003, 2004) investigated thermal post-buckling of rods and came up with a closed form solution via uncoupled elliptical integrals. Large thermal deflections for Timoshenko beams subjected transversely non-uniform temperature rise and with pinned-pinned as well as fixed-fixed ends were numerically analyzed by Li and Song (2006). Aristizabal-Ochao (2007) developed a new set of slope deflection equations for Timoshenko beam-columns which includes the combined effects of shear and bending deformations, and second-order axial load effects in a classical manner and emphasized the great importance of shear effects on static, tension and compression stability and dynamic behaviour of elastomeric bearings used for seismic isolation. Both thermal buckling and post-buckling of pinned–fixed beams resting on an elastic foundation were investigated by Song and Li (2007). Vaz and Nascimento (2007) examined a perturbation solution for the initial post-buckling of beams that were supported on an elastic foundation under uniform thermal load.

The large-deflection analysis and post-buckling behaviour of laterally braced or unbraced slender beam-columns of symmetrical cross section subjected to end loads (forces and moments) with both ends partially restrained against rotation, including the effects of out-of-plumbness and a new set of slope-deflection equations for Timoshenko beam–columns of symmetrical cross section with semi-rigid connections that include the combined effects of shear and bending deformations, and second-order axial load effects were developed in a classical manner by Aristizabal-Ochao (2008). Evandro and Joao (2008) investigated a simple and efficient methodology for sensitivity analysis of geometrically non-linear structures subjected to thermo-mechanical loading in regular and critical states.

Thermal post-buckling analysis of uniform, isotropic, slender and shear flexible columns were presented using a rigorous finite element formulation and a much simpler intuitive formulation by Gupta et al. (2009). Geometrically non-linear static analysis of a simply supported beam made of hyper elastic material subjected to a non-follower transversal uniformly distributed load was analyzed by Kocatürk and Akbaş (2010) using finite element model of the beam constructed by using total Lagrangian finite element model of two dimensional continuum for a twelve-node quadratic element.

Gupta et al. (2010a) investigated simple, elegant, and accurate closed-form expressions for predicting the post-buckling behaviour of composite beams with axially immovable ends using the Rayleigh-Ritz method. Thermal post-buckling analysis of columns with axially immovable ends were studied using the Rayleigh-Ritz method by Gupta et al. (2010b). Vaz et al. (2010) examined a perturbation solution for the initial post-buckling behaviour of slender beams that were supported assumed to be double-hinged with fixed ends, preventing thermal expansion.

The aim of this paper was to investigate the buckling and post-buckling responses of the considered beam made of hyper elastic material under uniform temperature rise.

As it is known, when two dimensions of a structural element is very small compared to the other dimension, then, for reducing the number of unknowns, one of the beam theories was used. When the dimensions of the considered element become close to each other, the beam theories lose accuracy and therefore they are not valid any more. According to assumptions made in these theories, some of the free boundary conditions can not be satisfied. However, in two dimensional solid continuum assumptions, only one dimension of the considered element is small compared to other dimensions. In the present study, every finite element of the beam was assumed as a two dimensional solid continuum which is a more realistic approach to the considered problem compared to beam theories.

The development of the formulations of general solution procedure of non-linear problems follows the general outline of the derivation given by Zienkiewichz and Taylor (2000). The geometrically non-linear responses of considered simply supported beam subjected to uniform thermal loading were obtained by using total Lagrangian finite element model of two-dimensional solid continuum. The TL finite element equations of two dimensional continuums for an eight-node quadratic element were used. These TL eight-node quadratic element formulations were given by Reddy (1993).

THEORY AND FORMULATIONS

A simply supported beam made of isotropic, hyper elastic material, with material or Lagrangian coordinate system $\left({}^{0}x_1,\ {}^{0}x_2,\ {}^{0}x_3\right)$ and with spatial or Euler coordinate system $\left({}^{2}x_1,\ {}^{2}x_2,\ {}^{2}x_3\right)$ having the origin *O was* shown in Figure 1. The supports of the beam were assumed to be pinned. The beam was subjected to a uniform temperature rise as seen in Figure 1.

While the derivation of the governing equations for most problems

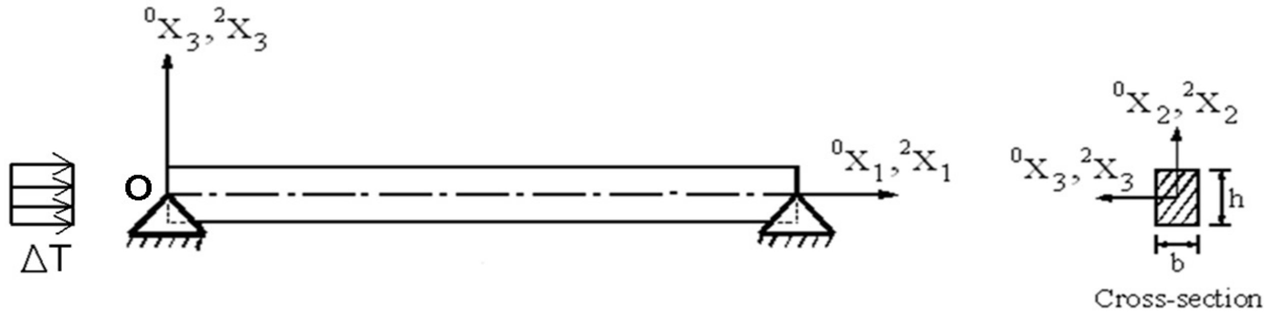

Figure 1. Pinned-pinned beam subjected to a uniform temperature rise.

was not unduly difficult, their solution by exact methods of analysis is a formidable task.

In such cases, numerical methods of analysis provide an alternative means of finding solutions. Numerical methods typically transform differential equations to algebraic equations that are to be solved using computers. The considered problem was a non linear one. Even linear problems may not admit exact solutions due to geometric and material complexities, but it was relatively easy to obtain approximate solutions using numerical methods (Reddy (2004).

There were some solutions for the special cases of boundary and loading conditions for large displacements of beams in the framework of Euler-Bernoulli beam theory. However, as far as the authors knew exact solution of a non-linear problem in the framework of two or three-dimensional, continuum approach was not possible. For the analysis of the pinned-pinned beam, the beam problem was considered as a two-dimensional continua problem: The total Lagrangian Finite element model of two dimensional continuums based on the total Lagrangian formulation for an eight-node quadratic element was used in the study.

For the solution of the total Lagrangian formulations of TL two dimensional continuum problems, small-step incremental approaches from known solutions were used. As it is known, it is possible to obtain solutions in a single increment of the external force only in the case of mild non-linearity (and no path dependence).

To obtain realistic solutions, physical insight into the nature of the problem and, usually, small-step incremental approaches from known solutions were essential. Such increments were always required if the constitutive law relating stress and strain changes was path dependent. Also, such incremental procedures were useful to reduce excessive numbers of iterations and in following the physically correct path.

In this study, small-step incremental approaches from known solutions with Newton-Raphson iteration method were used in which the solution for $n+1$ th load increment and i th iteration was obtained in the following form:

$$d\,u_n^i = \left(K_T^i\right)^{\mathbf{-1}} R_{n+1}^i \tag{1}$$

Where K_T^i is the stiffness matrix corresponding to a tangent direction at the i th iteration, du_n^i is the solution increment vector at the i th iteration and $n+1$ th load increment, R_{n+1}^i is the residual vector at the i th iteration and $n+1$ th load increment.

This iteration procedure was continued until the difference between two successive solution vectors were less than a selected tolerance criterion in Euclidean norm given by:

$$\sqrt{\frac{\left[\left(d\,u_n^{i+1} - d\,u_n^i\right)^T \left(d\,u_n^{i+1} - d\,u_n^i\right)\right]^2}{\left[\left(d\,u_n^{i+1}\right)^T \left(d\,u_n^{i+1}\right)\right]^2}} \le \zeta_{tol} \tag{2}$$

A series of successive approximations gave:

$$u_{n+1}^{i+1} = u_{n+1}^i + du_{n+1}^i = u_n + \Delta u_n^i \tag{3}$$

Where

$$\Delta u_n^i = \sum_{k=1}^{i} du_n^k \tag{4}$$

Total displacement fields and incremental displacement fields were expressed in terms of nodal displacements as follows:

$$\mathrm{u} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} \sum_{j=1}^{8} u_j \psi_j\left({}^o x_1, {}^o x_2\right) \\ \sum_{j=1}^{8} v_j \psi_j\left({}^o x_1, {}^o x_2\right) \end{Bmatrix} \tag{5}$$

$$\widehat{\mathrm{u}} = \begin{Bmatrix} \widehat{u} \\ \widehat{v} \end{Bmatrix} = \begin{Bmatrix} \sum_{j=1}^{8} \overline{u}_j \psi_j\left({}^o x_1, {}^o x_2\right) \\ \sum_{j=1}^{8} \overline{v}_j \psi_j\left({}^o x_1, {}^o x_2\right) \end{Bmatrix} \tag{6}$$

Where $\psi_j(x)$ are interpolation functions for a twelve-node quadratic element and can be found in Reddy (1993), $\overline{u}_j$ and $\overline{v}_j$

are the components of vectors of nodal displacements in the ${}^0x_1$ and ${}^0x_2$ directions respectively. The tangent stiffness matrix K_T^i and the residual vector R_{n+1}^i which are to be used in Equation 1 at the i th iteration for the total Lagrangian finite element model of a two dimensional continuum for an eight-node quadratic element were given below:

$$K_T^i \, du_n^i = R_{n+1}^i \qquad (7a)$$

$$\begin{bmatrix} K^{11L}+K^{11NL} & K^{12L} \\ K^{21L} & K^{22L}+K^{22NL} \end{bmatrix}^i \begin{Bmatrix} \bar{u} \\ \bar{v} \end{Bmatrix}^i = \begin{Bmatrix} {}_0^2F^1 - {}_0^1F^1 \\ {}_0^2F^2 - {}_0^1F^2 \end{Bmatrix}^i \qquad (7b)$$

The explicit expressions of K^{11L}, K^{11NL}, K^{12L}, K^{21L}, K^{22L}, K^{22NL}, ${}_0^1F^1$ and ${}_0^1F^2$ were given in Reddy (2004) and Kocatürk and Akbaş (2010).

$${}_0^2F_i^1 = h_e \int_{\Omega^e} {}_0^2f_{0_{x_1}} \psi_i d^0x_1 d^0x_1 + h_e \int_{\Gamma^e} {}_0^2t_{0_{x_1}} \psi_i d^0x_1 \qquad (8a)$$

$${}_0^2F_i^2 = h_e \int_{\Omega^e} {}_0^2f_{0_{x_2}} \psi_i d^0x_1 d^0x_1 + h_e \int_{\Gamma^e} {}_0^2t_{0_{x_2}} \psi_i d^0x_1 \qquad (8b)$$

Where ${}_0^2f_{0_{x_1}}$, ${}_0^2f_{0_{x_1}}$ are the body forces; ${}_0^2t_{0_{x_1}}$, ${}_0^2t_{0_{x_2}}$ are the surface forces in the ${}^0x_1$ and ${}^0x_2$ directions, respectively.

It is assumed that the temperature rise was uniform. Temperature rise from the undeformed state of beam was denoted by ΔT. The considered material was hyper elastic. In this case, the constitutive relation between the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor with a temperature rise can be assumed as follows:

$${}_0^1\mathbf{S} = \begin{Bmatrix} {}_0^1S_{11} \\ {}_0^1S_{22} \\ {}_0^1S_{12} \end{Bmatrix} = \begin{bmatrix} {}_0C_{11} & {}_0C_{12} & 0 \\ {}_0C_{12} & {}_0C_{22} & 0 \\ 0 & 0 & {}_0C_{66} \end{bmatrix} \begin{Bmatrix} {}_0^1E_{11} - \alpha_1 \Delta T \\ {}_0^1E_{22} - \alpha_2 \Delta T \\ 2\,{}_0^1E_{12} \end{Bmatrix} \qquad (9)$$

where ${}_0^1S_{11}$, ${}_0^1S_{22}$, ${}_0^1S_{12}$ are the components of the second Piola-Kirchhoff stress tensor components in the C_1 configuration of the body, ${}_0C_{ij}$ are the components of the reduced constitutive tensor in the C_0 configuration of the body, α_1 and α_2 are coefficients of thermal expansion in the ${}^0x_1$ and ${}^0x_2$ directions respectively. The components of the reduced constitutive tensor can be written in terms of Young modulus E and Poisson's ratio ν as follows:

$$\left.\begin{aligned} {}_0C_{11} &= \frac{E}{1-\nu^2}, \quad {}_0C_{12} = {}_0C_{21} = \frac{\nu E}{1-\nu^2}, \\ {}_0C_{22} &= \frac{E}{1-\nu^2}, \quad {}_0C_{66} = \frac{E}{2(1+\nu)} \end{aligned}\right\} \qquad (10)$$

The Green-Lagrange strain tensor's expression in terms of displacements in the case of two-dimensional solid continuum was given by Reddy (2004). Numerical calculations of the integrals in the rigidity matrices will be calculated by using five-point Gauss rule. The strains were assumed as small. It should be noted that the formulations given by Equations 5 to 10 are adopted from Reddy (2004).

RESULTS AND DISCUSSION

By use of the usual assembly process, the system tangent stiffness matrix given in Equation 1 was obtained by using the element stiffness matrixes given above for the total Lagrangian Finite element model of two dimensional continuum based on the total Lagrangian formulation for eight-node quadratic element. In the numerical integrations, five-point Gauss integration rule was used. The material of the beam was hyper elastic material and isotropic. Convergence analysis was performed for uniform thermal load for various numbers of finite elements in ${}^0x_1$ and ${}^0x_2$ directions. When the number of finite elements in ${}^0x_1$ direction is $m=40$ and when the number of elements in ${}^0x_2$ direction is $n=8$ for the total Lagrangian finite element model of two dimensional continuums for an eight-node quadratic element, the considered stresses and displacements converge perfectly.

In order to establish the accuracy of the present formulation and the computer program developed by the authors, results obtained from the present study were compared with the available results in the literature. For this purpose, thermal buckled configurations of pinned-pinned beam with different values of dimensionless uniform thermal load τ (Uniform temperature is non-dimensionalized by $\tau = \lambda^2 \alpha(\Delta T)$, where λ is slenderness of beam, for a beam with constant rectangular cross-section, the slenderness is $\lambda = 2\sqrt{3}\delta$, $\delta = L/h$ and α is the coefficient of thermal expansion.), are compared with data presented in Li and

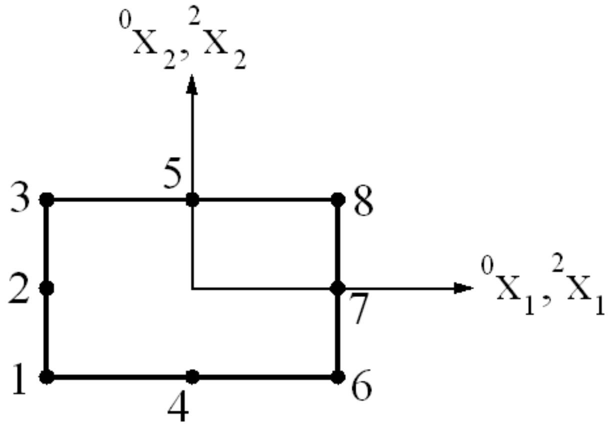

Figure 2. Eight-node quadratic plane element.

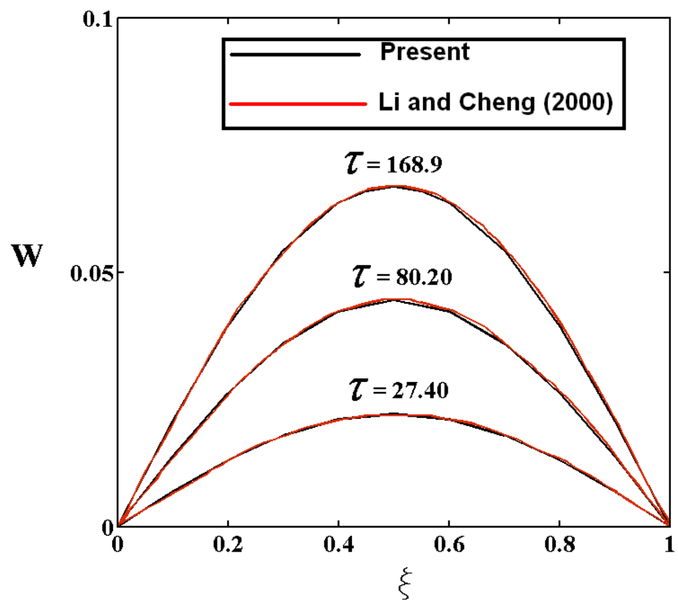

Figure 3. Thermal buckling configurations for some values of dimensionless uniform temperature rise τ for finite element model of two dimensional solid continuum for $\lambda = 120$.

Cheng (2000). For $\lambda = 120$, Figure 3 shows that the present results were in good agreement with Figure 2a of Li and Cheng (2000). In Figure 3, W is dimensionless of displacement and ξ is dimensionless of the beam length.

In Figure 4, the thermal buckled configurations of the axis of the beam for different values of τ and λ were given.

In Tables 1 and 2, dimensionless central deflections $f = W(0.5)$ and the left-end rotation angle $\phi = \varphi(0)$ (degree) were calculated respectively for various dimensionless uniform temperature rise τ and geometric parameter L/h for pinned-pinned beam. It was seen from Table 1 and 2 that, with increase in the geometric parameter L/h, central deflections and rotation angles decrease gradually.

In Table 3, dimensionless end constrained forces p in the horizontal direction $\left(p = \dfrac{L^2 P}{EI}\right)$ was calculated respectively for various dimensionless uniform temperature rise τ and geometric parameter L/h for pinned-pinned beam. It is known that the thermal buckling occurs when the temperature is greater than the critical temperature value, namely when $\tau > \tau_{cr}$. From Table 3, it can be seen that increase in the dimensionless uniform temperature rise τ, the magnitude of dimensionless end constrained forces p in the horizontal direction decreases gradually in lower percentage for $\tau > \tau_{cr}$.

Figure 5 showed the normal stress (Cauchy stress) distributions at the midpoint of the beam, for the thermal post-buckling case, and for some given ratios of L/h for uniform temperature $\Delta T = 300^{\circ}$, for Young's modulus $E = 21000000 \text{ N/cm}^2$ and Poisson's ratio $\nu = 0.2875$. The normal stresses at the midpoint of the beam were obtained on the cross section. Also, Figure 5 showed that, with decrease in the ratio of L/h, the Cauchy normal stresses increase.

For comparison of the dimensionless central deflections $f = W(0.5)$ and left-end rotational angle $\phi = \varphi(0)$ (degree) versus uniform temperature rise τ for some values of L/h ratios at pinned-pinned beam, the obtained results were compared with those of Li and Song (2006) by inserting the material properties used in this reference under uniform temperature rise. In this study of Li and Song (2006), was used Timoshenko beam theory, the ratio of the elasticity modulus (E) and shear modulus (G) was taken as $E/G = 206/80$ and the uniform temperature rise are non-dimensionalized by $\tau = \lambda^2 \alpha(\Delta T)$, where λ is slenderness of beam. For a beam with constant rectangular cross-section, the slenderness is $\lambda = 2\sqrt{3}\delta, \delta = L/h$. α is the coefficient of thermal expansion. Figures 6 and 7 showed the between results of finite element model of two dimensional solid continuum and Timoshenko beam theory that was obtained by Figures 1a and Figure 2 of Li and Song

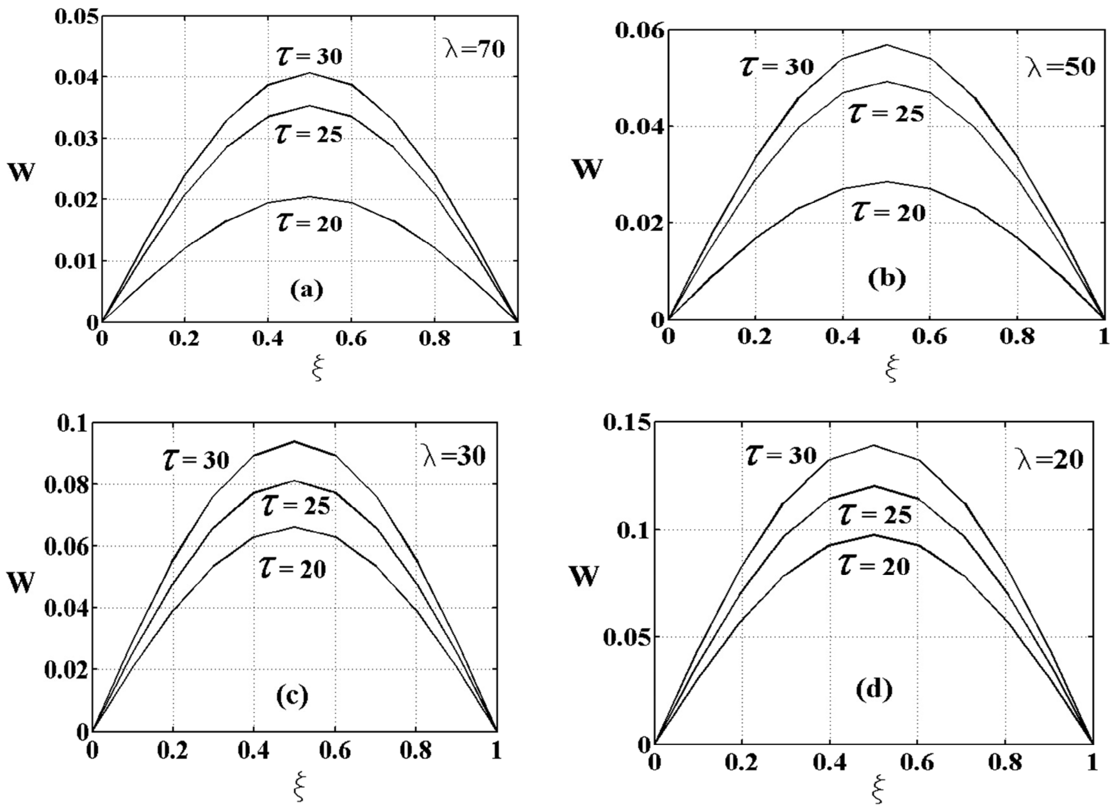

Figure 4. Thermal buckling configuration for some values dimensionless uniform temperature rise τ . a) $\lambda = 70$, b) $\lambda = 50$,c) $\lambda = 30$ and d) $\lambda = 20$.

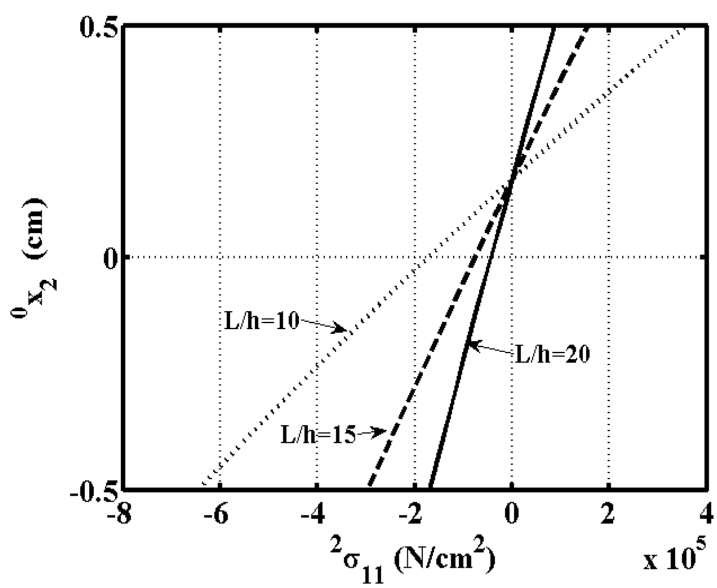

Figure 5. Cauchy normal stresses for the cross section at the the midpoint of the pinned-pinned beam for some given ratios of *L/h* for uniform temperature rise $\Delta T = 300^{\circ}$.

Table 1. Variation of the dimensionless central deflections $f = W(0.5)$ with dimen-sionless uniform temperature rise τ and geometric parameter L/h for pinned-pinned beam.

	L/h			
τ	**8**	**12**	**16**	**20**
12	0.0306	0.0215	0.0163	0.0132
15	0.0499	0.0340	0.0257	0.0207
20	0.0713	0.0482	0.0362	0.0291
25	0.0876	0.0590	0.0444	0.0356
30	0.1013	0.0680	0.0512	0.0411

Table 2. Variation of the left-end rotation angle $\phi = \varphi(0)$ (degree) with dimensionless uniform temperature rise τ and geometric parameter L/h for pinned-pinned beam.

	L/h			
τ	**8**	**12**	**16**	**20**
12	5.2414	3.7877	2.8950	2.3635
15	8.4777	5.9802	4.5570	3.6935
20	11.9300	8.4043	6.4136	5.1916
25	14.4458	10.2232	7.8209	6.3341
30	16.4710	11.6937	8.9928	7.2902

Table 3. Variation of the dimensionless end constraint force p in the horizontal direction with dimensionless uniform temperature rise τ and geometric parameter L/h for pinned-pinned beam.

	L/h			
τ	**8**	**12**	**16**	**20**
12	9.5170	9.7048	9.7811	9.8274
15	9.4993	9.6988	9.7761	9.8242
20	9.4830	9.6892	9.7488	9.7994
25	9.4888	9.6787	9.7626	9.8156
30	9.4252	9.6676	9.7519	9.8112

(2006). It is clearly seen from Figure 6 and 7 that, with decrease in the $\delta = L/h$, the difference between the results of finite element model of two dimensional solid continuum and Timoshenko beam theory differs considerably.

Conclusion

Post-buckling analysis of a simply supported beam subjected to a uniform thermal loading has been studied. In this study, the finite element model of the beam was constructed by using total Lagrangian finite element model of two dimensional continuum for an eight-node quadratic element. The considered highly non-linear problem was solved by using incremental displacement-based finite element method in conjunction with Newton-Raphson iteration method. There was no restriction on the displacements. The comparison was performed. It was seen from the investigations that the difference between the results of finite element model of two dimensional solid continuum and Timoshenko beam theory increases considerably while the beam with small slenderness ratio decreases. Therefore, for small slenderness of beam, finite element model of two dimensional solid continuums must be used instead of Timoshenko

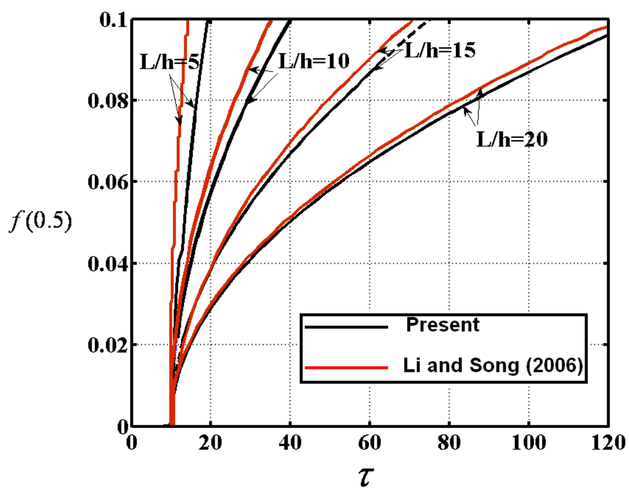

Figure 6. Dimensionless central deflection $f = W(0.5)$ versus uniform temperature rise τ for some given ratios of L/h for pinned-pinned beam.

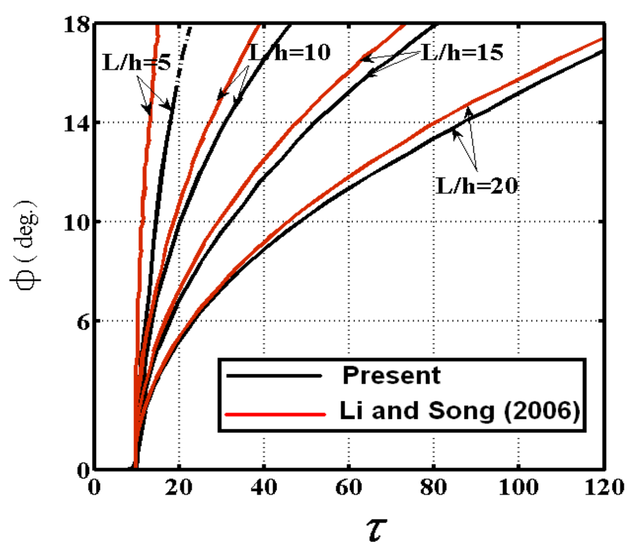

Figure 7. Left-end rotational angle $\phi = \varphi(0)$ (degree) versus uniform temperature rise τ for some given ratios of L/h for pinned-pinned beam.

beam theory.

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