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Full Length Research Paper

Double traveling wave solutions for some nonlinear partial differential equations in mathematical Physics

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In this article, we construct the double soliton exact solutions for some nonlinear partial differential equations in mathematical physics via (2+1)-dimensional Painleve integrable Burgers equations, (2+1)-dimensional breaking soliton equations and (2+1)-dimensional Nizhnik-Novikov Veselov equations. We obtain many new types of complexiton soliton solution, that is, various combination of trigonometric periodic function and hyperbolic function, various combination of trigonometric periodic function and rational function, various combination of hyperbolic function and rational function solutions.

Key words: Double soliton solution, the traveling wave solution, the (2+1)-dimensional Painleve integrable Burgers equations, the (2+1)-dimensional breaking soliton equations and the (2+1)-dimensional Nizhnik-Novikov Veselov equations.

INTRODUCTION

Nonlinear partial differential equations are known to describe a wide variety of phenomena not only in physics, where applications extend over magneto fluid dynamics, water surface gravity waves, electromagnetic radiation reactions, and ion acoustic waves in plasma, just to name a few, but also in biology, chemistry and several other fields. One of the important tasks in the study of nonlinear partial differential equations is to seek exact and explicit solutions. Many powerful methods have been presented by authors such as the inverse scattering transform (Ablowitz and Clarkson, 1991), the Bäcklund transform (Chen et al., 2002; Yan, 2003), the generalized Riccati equation (Chen et al., 2010; Wang and Zhang, 2007), the Jacobi elliptic expansion (Chen and Zhang, 2004; Liu et al., 2001), the extended tanh-function method (Chen and Zhang, 2004; Abdou, 2007; Fan, 2000;), the F-expansion method (Abdou, 2008; Yomba, 2010), the exp-function expansion method (Hua, 2010; He and Wu, 2006), the sub-ODE method (Wang et al., 2007; Li and Wang, 2007), the homogeneous balance method (Wang, 1995), the extended sine-cosine methods(Yan and Zhang, 2004), the complex hyperbolic function method (Zhang and Wang, 2004), the (G'/G) -expansion method (Wang et al., 2008; Zayed and Gepreel, 2008; Lingxiao et al., 2010) and so on. Recently, the extended coupled sub-equations expansion method as the extension of multiple Riccati equations expansion method is efficiently applied by many researchers to a great variety of NLPDEs (Gepreel, 2011; Zayed and Gepreel, 2011). The main objective of this article is to construct the double soliton exact traveling wave solutions for some nonlinear PDE's by using the extended coupled (G'/G) and (H'/H) expansion function method, where G and H satisfy the second-order linear ordinary differential equations. Many new kinds of double soliton exact solutions are obtained.

DESCRIPTION OF THE DOUBLE SOLITON SOLUTIONS METHOD FOR SOME NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS (PDE'S)

Consider the nonlinear PDE's in the following form:

$$F(u,u_t,u_x,u_y,u_{xx},u_{xxy},...)=0, \tag{1}$$

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where $u = u(x, y, t)$ is unknown function, and F is a polynomial in $u(x, y, t)$ and its partial derivatives. We would like to outline the main steps of the double soliton solutions method for nonlinear PDE's as follows:

Step 1

We suppose the traveling wave transformation

$$u(x, y, t) = u(\xi, \eta), \tag{2}$$

where

$$\xi = M_1 x + K_1 t + L_1 y, \qquad \eta = M_2 x + K_2 t + L_2 y, \tag{3}$$

and M_1, M_2, L_1, L_2, K_1 and K_2 are arbitrary constants to be determined later.

Step 2

We introduce a more Ansatz in term of a finite form expansion in the following forms:

$$u(\xi, \eta) = a_0 + \sum_{k=1}^{N} \left(\sum_{i+j=k} a_i^j \left(\frac{G'(\xi)}{G(\xi)} \right)^i \left(\frac{H'(\eta)}{H(\eta)} \right)^j \right), \tag{4}$$

where $G(\xi)$ and $H(\eta)$ satisfy the following second-order linear ODEs:

$$\begin{aligned} &G''(\xi) + \lambda_1 G'(\xi) + \mu_1 G(\xi) = 0, \\ &H''(\eta) + \lambda_2 H'(\eta) + \mu_2 H(\eta) = 0 \end{aligned} \tag{5}$$

and $a_0, a_i^j (i, j = 0,1,..., N), \lambda_1, \lambda_2, \mu_1, \mu_2$ are constants to be determined.

Step 3

The positive integer " N " can be determined by balancing the highest-order derivatives with the nonlinear terms appearing in Equation (1).

Step 4

Substituting Equation (4) into Equation (1) and using Equation 5, collecting all terms with the same powers of $(G'/G)^i$ and $(H'/H)^i (i = 0,1,2,...)$ together. Setting each coefficients of this polynomial to be zero, yields a set of algebraic equations for $a_0, a_i^j (i, j = 0,1,..., N), \lambda_1, \lambda_2, \mu_1, \mu_2, M_1, M_2, L_1, L_2, K_1$ and K_2.

Step 5

Solve the over-determined system of nonlinear algebraic equations by using the symbolic computation as Maple or Mathematica. We end up with explicit expressions for $a_0, a_i^j (i, j = 0,1,..., N), \lambda_1, \lambda_2, \mu_1, \mu_2, M_1, M_2, L_1, L_2, K_1$ and K_2.

Step 6

Since the general solutions of Equation 5 have been known for us, and substituting $a_0, a_i^j (i, j = 0,1,..., N), \lambda_1, \lambda_2, \mu_1, \mu_2, M_1, M_2, L_1, L_2, K_1, K_2$ and the general solutions of 5 into 4, we have determined the different types of double soliton exact solutions for the nonlinear partial differential Equation (1). It is well known that the general solutions of Equation (5) are listed as follows:

(i) When $\lambda_1^2 - 4\mu_1 > 0$ and $\lambda_2^2 - 4\mu_2 > 0$, then

$$\begin{aligned} \frac{G'(\xi)}{G(\xi)} &= \left[\frac{1}{2}\sqrt{\lambda_1^2 - 4\mu_1} \left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi)} \right) - \frac{\lambda_1}{2} \right], \\ \frac{H'(\eta)}{H(\eta)} &= \left[\frac{1}{2}\sqrt{\lambda_2^2 - 4\mu_2} \left(\frac{C\cosh(\frac{1}{2}\sqrt{(\lambda_2^2 - 4\mu_2)}\ \eta) + D\sinh(\frac{1}{2}\sqrt{(\lambda_2^2 - 4\mu_2)}\ \eta)}{C\sinh(\frac{1}{2}\sqrt{(\lambda_2^2 - 4\mu_2)}\ \eta) + D\cosh(\frac{1}{2}\sqrt{(\lambda_2^2 - 4\mu_2)}\ \eta)} \right) - \frac{\lambda_2}{2} \right], \end{aligned} \tag{6}$$

(ii) When $\lambda_1^2 - 4\mu_1 > 0$ and $\lambda_2^2 - 4\mu_2 < 0$, then

$$\frac{G'(\xi)}{G(\xi)} = \left[\frac{1}{2}\sqrt{\lambda_1^2 - 4\mu_1} \left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi)} \right) - \frac{\lambda_1}{2} \right],$$

$$\frac{H'(\eta)}{H(\eta)} = \left[\frac{1}{2}\sqrt{4\mu_2 - \lambda_2^2} \left(\frac{-C\sin(\frac{1}{2}\sqrt{(4\mu_2 - \lambda_2^2)}\ \eta) + D\cos(\frac{1}{2}\sqrt{(4\mu_2 - \lambda_2^2)}\ \eta)}{C\cos(\frac{1}{2}\sqrt{(4\mu_2 - \lambda_2^2)}\ \eta) + D\sin(\frac{1}{2}\sqrt{(4\mu_2 - \lambda_2^2)}\ \eta)} \right) - \frac{\lambda_2}{2} \right], \quad (7)$$

(iii) When $\lambda_1^2 - 4\mu_1 < 0$ and $\lambda_2^2 - 4\mu_2 < 0,$ then

$$\frac{G'(\xi)}{G(\xi)} = \left[\frac{1}{2}\sqrt{4\mu_1 - \lambda_1^2} \left(\frac{-A\sin(\frac{1}{2}\sqrt{(4\mu_1 - \lambda_1^2)}\ \xi) + B\cos(\frac{1}{2}\sqrt{(4\mu_1 - \lambda_1^2)}\ \xi)}{A\cos(\frac{1}{2}\sqrt{(4\mu_1 - \lambda_1^2)}\ \xi) + B\sin(\frac{1}{2}\sqrt{(4\mu_1 - \lambda_1^2)}\ \xi)} \right) - \frac{\lambda_1}{2} \right],$$

$$\frac{H'(\eta)}{H(\eta)} = \left[\frac{1}{2}\sqrt{4\mu_2 - \lambda_2^2} \left(\frac{-C\sin(\frac{1}{2}\sqrt{(4\mu_2 - \lambda_2^2)}\ \eta) + D\cos(\frac{1}{2}\sqrt{(4\mu_2 - \lambda_2^2)}\ \eta)}{C\cos(\frac{1}{2}\sqrt{(4\mu_2 - \lambda_2^2)}\ \eta) + D\sin(\frac{1}{2}\sqrt{(4\mu_2 - \lambda_2^2)}\ \eta)} \right) - \frac{\lambda_2}{2} \right], \quad (8)$$

(iv) When $\lambda_1^2 - 4\mu_1 > 0$ and $\lambda_2^2 - 4\mu_2 = 0,$ then

$$\frac{G'(\xi)}{G(\xi)} = \left[\frac{1}{2}\sqrt{\lambda_1^2 - 4\mu_1} \left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi)} \right) - \frac{\lambda_1}{2} \right],$$

$$\frac{H'(\eta)}{H(\eta)} = \left[\frac{D}{C + D\eta} - \frac{\lambda_2}{2} \right], \quad (9)$$

(v) When $\lambda_1^2 - 4\mu_1 < 0$ and $\lambda_2^2 - 4\mu_2 = 0,$ then

$$\frac{G'(\xi)}{G(\xi)} = \left[\frac{1}{2}\sqrt{4\mu_1 - \lambda_1^2} \left(\frac{-A\sin(\frac{1}{2}\sqrt{(4\mu_1 - \lambda_1^2)}\ \xi) + B\cos(\frac{1}{2}\sqrt{(4\mu_1 - \lambda_1^2)}\ \xi)}{A\cos(\frac{1}{2}\sqrt{(4\mu_1 - \lambda_1^2)}\ \xi) + B\sin(\frac{1}{2}\sqrt{(4\mu_1 - \lambda_1^2)}\ \xi)} \right) - \frac{\lambda_1}{2} \right],$$

$$\frac{H'(\eta)}{H(\eta)} = \left[\frac{D}{C + D\eta} - \frac{\lambda_2}{2} \right], \quad (10)$$

(vi) When $\lambda_1^2 - 4\mu_1 = 0$ and $\lambda_2^2 - 4\mu_2 = 0,$ then

$$\frac{G'(\xi)}{G(\xi)} = \left[\frac{B}{A + B\xi} - \frac{\lambda_1}{2} \right],$$

$$\frac{H'(\eta)}{H(\eta)} = \left[\frac{D}{C + D\eta} - \frac{\lambda_2}{2} \right],$$

where A, B, C and D are arbitrary constants.

APPLICATIONS

In this section, we will apply the proposed double soliton solutions method for some nonlinear partial differential equations namely; the (2+1)-dimensional Painleve integrable Burgers equations, the (2+1)-dimensional breaking soliton equations and the (2+1)-dimensional Nizhnik-Novikov Veselov quations, which play an important role in the mathematical physics and have been concerned with many researchers.

Example 1: The (2+1)-dimensional Painleve integrable Burgers equations

We start with the (2+1)-dimensional Painleve integrable Burgers equations in the following form (Zayed and Gepreel, 2009):

$$-u_t + uu_y + \alpha v u_x + \beta u_{yy} + \alpha\beta u_{xx} = 0,$$
$$u_x + v_y = 0, \quad (11)$$

where α and β are arbitrary constants.

We suppose the traveling wave transformations in the following form:

$$u(x,y,t)=u(\xi,\eta), \qquad v(x,y,t)=v(\xi,\eta), \tag{12}$$

where

$$\xi=M_1x+K_1t+L_1y, \qquad \eta=M_2x+K_2t+L_2y, \tag{13}$$

and $M_1, M_2, L_1, L_2, K_1, K_2$ are arbitrary constants to be determined later.

The solutions of Equation (11) can be expressed by a polynomial in (G'/G) and (H'/H) as the following form:

$$u(\xi,\eta)=a_0+\sum_{k=1}^{N}\left(\sum_{i+j=k} a_i^j\left(\frac{G'(\xi)}{G(\xi)}\right)^i\left(\frac{H'(\eta)}{H(\eta)}\right)^j\right), \tag{14}$$

$$v(\xi,\eta)=b_0+\sum_{k=1}^{M}\left(\sum_{i+j=k} b_i^j\left(\frac{G'(\xi)}{G(\xi)}\right)^i\left(\frac{H'(\eta)}{H(\eta)}\right)^j\right), \tag{15}$$

where a_0, b_0, $a_i^j(i,j=1,...,N)$ and $b_i^j(i,j=1,...,M)$ are arbitrary constants, while $G(\xi)$ and $H(\eta)$ satisfy the second-order linear ODE's (5). Considering the homogeneous balance between the highest-order derivatives and nonlinear terms in Equation (11) we have $N=M=1$, and consequently we can rewrite Equations (14) and (15) in the following form:

$$u(\xi,\eta)=a_0+a_1\left(\frac{G'(\xi)}{G(\xi)}\right)+a_2\left(\frac{H'(\eta)}{H(\eta)}\right), \tag{16}$$

$$v(\xi,\eta)=b_0+b_1\left(\frac{G'(\xi)}{G(\xi)}\right)+b_2\left(\frac{H'(\eta)}{H(\eta)}\right), \tag{17}$$

where a_0, a_1, a_2, b_0, b_1 and b_2 are arbitrary constants to be determined. With the aid of Maple, we substitute Equation (16), (17) along with (5) into Equation (11) and set the coefficients of $(G'/G)^i$ and $(H'/H)^i$ to be zero, yield a set of over determined algebraic equations with respect to $a_i, b_i (i=0,1,2), \lambda_1, \lambda_2, \mu_1, \mu_2, M_1, M_2, L_1, L_2, K_1$ and K_2. On using the Maple software package, we solve the over determined algebraic equations and consequently we get the following results:

$$\begin{aligned}
&M_1=\frac{b_1L_1}{a_1}, \qquad M_2=-\frac{a_1L_2}{\alpha b_1}, \qquad \beta=\frac{a_1}{2L_1},\\
&a_0=\frac{2L_2K_1a_1^2+(L_1\lambda_1+L_2\lambda_2)(a_1L_2b_1^2\alpha+a_1^3L_2)+2\alpha b_1^2K_2L_1}{2L_2(\alpha b_1^2+a_1^2)L_1},\\
&b_0=-\frac{\left[-\alpha^2b_1^4L_2L_1\lambda_1+a_1^2\alpha b_1^2L_2(L_2\lambda_2-L_1\lambda_1)+a_1^4L_2^2\lambda_2+2a_1\alpha b_1^2(K_2L_1-L_2K_1)\right]}{2\alpha b_1L_2L_1(\alpha b_1^2+a_1^2)},\\
&b_2=-\frac{a_1^2L_2}{\alpha b_1L_1}, \qquad a_2=\frac{a_1L_2}{L_1},
\end{aligned} \tag{18}$$

where $a_1, b_1, \alpha, \lambda_1, \lambda_2, L_1L_2, \mu_1, \mu_2, K_1$ and K_2 are arbitrary constants. Substituting Equation (18) into Equations (16) and (17), we get:

$$\begin{aligned}
u(\xi,\eta)=&\frac{2L_2K_1a_1^2+(L_1\lambda_1+L_2\lambda_2)(a_1L_2b_1^2\alpha+a_1^3L_2)+2\alpha b_1^2K_2L_1}{2L_2(\alpha b_1^2+a_1^2)L_1}\\
&+a_1\left(\frac{G'(\xi)}{G(\xi)}\right)+\frac{a_1L_2}{L_1}\left(\frac{H'(\eta)}{H(\eta)}\right),
\end{aligned} \tag{19}$$

$$v(\xi,\eta) = -\frac{\left[-\alpha^2 b_1^4 L_2 L_1 \lambda_1 + a_1^2 \alpha b_1^2 L_2 (L_2\lambda_2 - L_1\lambda_1) + a_1^4 L_2^2 \lambda_2 + 2a_1 \alpha b_1^2 (K_2 L_1 - L_2 K_1)\right]}{2\alpha b_1 L_2 L_1 (\alpha b_1^2 + a_1^2)} + b_1\left(\frac{G'(\xi)}{G(\xi)}\right) - \frac{a_1^2 L_2}{\alpha b_1 L_1}\left(\frac{H'(\eta)}{H(\eta)}\right), \tag{20}$$

Where

$$\xi = \frac{b_1 L_1}{a_1} x + K_1 t + L_1 y, \qquad \eta = -\frac{a_1 L_2}{\alpha b_1} x + K_2 t + L_2 y. \tag{21}$$

We have the following families of exact solutions:

Family 1

If $\lambda_1^2 - 4\mu_1 > 0$ and $\lambda_2^2 - 4\mu_2 > 0,$ we get the double soliton solutions of Equation (11) which have the following form:

$$\begin{aligned}
u(\xi,\eta) &= \frac{L_2 K_1 a_1^2 + \alpha b_1^2 K_2 L_1}{L_2(\alpha b_1^2 + a_1^2) L_1} \\
&+ \frac{a_1\sqrt{\lambda_1^2 - 4\mu_1}}{2}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\,\xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\,\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\,\xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\,\xi)}\right) \\
&+ \frac{a_1 L_2\sqrt{\lambda_2^2 - 4\mu_2}}{2L_1}\left(\frac{C\cosh(\frac{1}{2}\sqrt{(\lambda_2^2 - 4\mu_2)}\,\eta) + D\sinh(\frac{1}{2}\sqrt{(\lambda_2^2 - 4\mu_2)}\,\eta)}{C\sinh(\frac{1}{2}\sqrt{(\lambda_2^2 - 4\mu_2)}\,\eta) + D\cosh(\frac{1}{2}\sqrt{(\lambda_2^2 - 4\mu_2)}\,\eta)}\right), \\
v(\xi,\eta) &= \frac{a_1 b_1(-K_2 L_1 + L_2 K_1)}{L_2 L_1(\alpha b_1^2 + a_1^2)} \\
&+ \frac{b_1\sqrt{\lambda_1^2 - 4\mu_1}}{2}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\,\xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\,\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\,\xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\,\xi)}\right) \\
&- \frac{a_1^2 L_2\sqrt{\lambda_2^2 - 4\mu_2}}{2\alpha b_1 L_1}\left(\frac{C\cosh(\frac{1}{2}\sqrt{(\lambda_2^2 - 4\mu_2)}\,\eta) + D\sinh(\frac{1}{2}\sqrt{(\lambda_2^2 - 4\mu_2)}\,\eta)}{C\sinh(\frac{1}{2}\sqrt{(\lambda_2^2 - 4\mu_2)}\,\eta) + D\cosh(\frac{1}{2}\sqrt{(\lambda_2^2 - 4\mu_2)}\,\eta)}\right).
\end{aligned} \tag{22}$$

Family 2

If $\lambda_1^2 - 4\mu_1 > 0,$ and $\lambda_2^2 - 4\mu_2 < 0,$ we get the complexiton soliton solutions of Equation (11) which have the following form:

$$u(\xi,\eta) = \frac{L_2 K_1 a_1^2 + \alpha b_1^2 K_2 L_1}{L_2(\alpha b_1^2 + a_1^2)L_1} + \frac{a_1\sqrt{\lambda_1^2 - 4\mu_1}}{2}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi)}\right) + \frac{a_1 L_2\sqrt{4\mu_2 - \lambda_2^2}}{2L_1}\left(\frac{-C\sin(\frac{1}{2}\sqrt{(4\mu_2 - \lambda_2^2)}\ \eta) + D\cos(\frac{1}{2}\sqrt{(4\mu_2 - \lambda_2^2)}\ \eta)}{C\cos(\frac{1}{2}\sqrt{(4\mu_2 - \lambda_2^2)}\ \eta) + D\sin(\frac{1}{2}\sqrt{(4\mu_2 - \lambda_2^2)}\ \eta)}\right),$$

$$v(\xi,\eta) = \frac{a_1 b_1(-K_2 L_1 + L_2 K_1)}{L_2 L_1(\alpha b_1^2 + a_1^2)} + \frac{b_1\sqrt{\lambda_1^2 - 4\mu_1}}{2}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi)}\right) - \frac{a_1^2 L_2\sqrt{4\mu_2 - \lambda_2^2}}{2\alpha b_1 L_1}\left(\frac{-C\sin(\frac{1}{2}\sqrt{(4\mu_2 - \lambda_2^2)}\ \eta) + D\cos(\frac{1}{2}\sqrt{(4\mu_2 - \lambda_2^2)}\ \eta)}{C\cos(\frac{1}{2}\sqrt{(4\mu_2 - \lambda_2^2)}\ \eta) + D\sin(\frac{1}{2}\sqrt{(4\mu_2 - \lambda_2^2)}\ \eta)}\right). \tag{23}$$

Family 3

If $\lambda_1^2 - 4\mu_1 > 0$, and $\lambda_2^2 - 4\mu_2 = 0$, we get:

$$u(\xi,\eta) = \frac{L_2 K_1 a_1^2 + \alpha b_1^2 K_2 L_1}{L_2(\alpha b_1^2 + a_1^2)L_1} + \frac{a_1 L_2}{L_1}\left(\frac{D}{C + D\eta}\right) + \frac{a_1\sqrt{\lambda_1^2 - 4\mu_1}}{2}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi)}\right),$$

$$v(\xi,\eta) = \frac{a_1 b_1(-K_2 L_1 + L_2 K_1)}{L_2 L_1(\alpha b_1^2 + a_1^2)} - \frac{a_1^2 L_2}{\alpha b_1 L_1}\left(\frac{D}{C + D\eta}\right) + \frac{b_1\sqrt{\lambda_1^2 - 4\mu_1}}{2}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2 - 4\mu_1)}\ \xi)}\right). \tag{24}$$

Family 4

If $\lambda_1^2 - 4\mu_1 < 0$, and $\lambda_2^2 - 4\mu_2 < 0$, we get the double triangular function solutions of Equation (11) which have the following form:

$$u(\xi,\eta)=\frac{L_2K_1a_1^2+\alpha b_1^2K_2L_1}{L_2(\alpha b_1^2+a_1^2)L_1}$$
$$+\frac{a_1\sqrt{4\mu_1-\lambda_1^2}}{2}\left(\frac{-A\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}{A\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}\right)$$
$$+\frac{a_1L_2\sqrt{4\mu_2-\lambda_2^2}}{2L_1}\left(\frac{-C\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}{C\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}\right),$$
$$v(\xi,\eta)=\frac{a_1b_1(-K_2L_1+L_2K_1)}{L_2L_1(\alpha b_1^2+a_1^2)}$$
$$+\frac{b_1\sqrt{4\mu_1-\lambda_1^2}}{2}\left(\frac{-A\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}{A\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}\right)$$
$$-\frac{a_1^2L_2\sqrt{4\mu_2-\lambda_2^2}}{2\alpha b_1L_1}\left(\frac{-C\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}{C\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}\right). \quad (25)$$

Family 5

If $\lambda_1^2-4\mu_1<0$, and $\lambda_2^2-4\mu_2=0$, we get:

$$u(\xi,\eta)=\frac{L_2K_1a_1^2+\alpha b_1^2K_2L_1}{L_2(\alpha b_1^2+a_1^2)L_1}+\frac{a_1L_2}{L_1}\left(\frac{D}{C+D\eta}\right)$$
$$+\frac{a_1\sqrt{4\mu_1-\lambda_1^2}}{2}\left(\frac{-A\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}{A\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}\right),$$
$$v(\xi,\eta)=\frac{a_1b_1(-K_2L_1+L_2K_1)}{L_2L_1(\alpha b_1^2+a_1^2)}-\frac{a_1^2L_2}{\alpha b_1L_1}\left(\frac{D}{C+D\eta}\right)$$
$$+\frac{b_1\sqrt{4\mu_1-\lambda_1^2}}{2}\left(\frac{-A\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}{A\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}\right). \quad (26)$$

Family 6

If $\lambda_1^2-4\mu_1=0$, and $\lambda_2^2-4\mu_2=0$, we get:

$$u(\xi,\eta)=\frac{L_2K_1a_1^2+\alpha b_1^2K_2L_1}{L_2(\alpha b_1^2+a_1^2)L_1}+a_1\left(\frac{B}{A+B\xi}\right)+\frac{a_1L_2}{L_1}\left(\frac{D}{C+D\eta}\right),$$
$$v(\xi,\eta)=\frac{a_1b_1(-K_2L_1+L_2K_1)}{L_2L_1(\alpha b_1^2+a_1^2)}+b_1\left(\frac{B}{A+B\xi}\right)-\frac{a_1^2L_2}{\alpha b_1L_1}\left(\frac{D}{C+D\eta}\right). \quad (27)$$

We should point out that not only the Equations (22) to (27) are the solutions obtained in this example, but also we have some new solutions corresponding to double solitary like wave solutions, double trigonometric function solutions and complexiton soliton solutions of the (2+1)-dimensional Painleve integrable Burgers equations, which are omitted here for simplicity .

Example 2: The (2+1)-dimensional breaking soliton equations

In this subsection, we study the (2 +1)-dimensional Breaking soliton equations in the following form (Zhang, 2007):

$$u_t + bu_{xxy} + 4buv_x + 4bu_x v = 0, \qquad u_y - v_x = 0, \tag{28}$$

where b is an arbitrary constant.

We suppose the traveling wave transformations in the following form:

$$u(x,y,t) = u(\xi,\eta), \qquad v(x,y,t) = v(\xi,\eta), \tag{29}$$

where

$$\xi = M_1 x + K_1 t + L_1 y, \qquad \eta = M_2 x + K_2 t + L_2 y, \tag{30}$$

and $M_1, M_2, L_1, L_2, K_1, K_2$ are arbitrary constants to be determined later.

The solutions of Equation (28) can be expressed by a polynomial in (G'/G) and (H'/H) as the following form:

$$u(\xi,\eta) = a_0 + \sum_{k=1}^{N}\left(\sum_{i+j=k} a_i^j \left(\frac{G'(\xi)}{G(\xi)}\right)^i \left(\frac{H'(\eta)}{H(\eta)}\right)^j\right), \tag{31}$$

$$v(\xi,\eta) = b_0 + \sum_{k=1}^{M}\left(\sum_{i+j=k} b_i^j \left(\frac{G'(\xi)}{G(\xi)}\right)^i \left(\frac{H'(\eta)}{H(\eta)}\right)^j\right), \tag{32}$$

where $a_0, b_0, a_i^j, (i,j=1,...,N)$ and $b_i^j (i,j=1,...,M)$ are arbitrary constants, while $G(\xi)$ and $H(\eta)$ satisfy the second order linear ODE's 5. Considering the homogeneous balance between the highest-order derivatives and nonlinear terms in Equation (28) we have $N = M = 2$, and consequently we can rewrite Equations (31) and (32) in the following form:

$$u(\xi,\eta) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 + a_3\left(\frac{H'(\eta)}{H(\eta)}\right) + a_4\left(\frac{H'(\eta)}{H(\eta)}\right)^2 + a_5\left(\frac{G'(\xi)}{G(\xi)}\right)\left(\frac{H'(\eta)}{H(\eta)}\right), \tag{33}$$

$$v(\xi,\eta) = b_0 + b_1\left(\frac{G'(\xi)}{G(\xi)}\right) + b_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 + b_3\left(\frac{H'(\eta)}{H(\eta)}\right) + b_4\left(\frac{H'(\eta)}{H(\eta)}\right)^2 + b_5\left(\frac{G'(\xi)}{G(\xi)}\right)\left(\frac{H'(\eta)}{H(\eta)}\right), \tag{34}$$

where $a_i, b_i (i = 0,1,...,5)$ are arbitrary constants to be determined. With the aid of Maple, we substitute (33), (34) along with (5) into Equation (28) and set the coefficients of $(G'/G)^i$ and $(H'/H)^i$ to be zero, yield a set of over determined algebraic equations with respect to $a_i, b_i (i = 0,1,...,5), \lambda_1, \lambda_2, \mu_1, \mu_2, M_1, M_2, L_1, L_2, K_1$ and K_2. On using the Maple software package, we solve the over determined algebraic equations, and consequently we get the following results:

$$\begin{aligned}
&a_0 = -\frac{\left[M_1^3 bL_2(\lambda_1^2 + 8\mu_1) - M_2(K_1 + 4bb_0 M_1)\right]}{4bM_1L_2}, \\
&a_1 = -\frac{3M_1^2\lambda_1}{2}, \qquad a_2 = -\frac{3M_1^2}{2}, \qquad a_3 = -\frac{3M_2^2\lambda_2}{2}, \\
&a_4 = -\frac{3M_2^2}{2}, \qquad b_1 = \frac{3M_1^2\lambda_1 L_2}{2M_2}, \qquad b_2 = \frac{3M_1^2 L_2}{2M_2}, \\
&b_3 = -\frac{3M_2\lambda_2 L_2}{2}, \qquad b_4 = -\frac{3M_2 L_2}{2}, \qquad L_1 = -\frac{M_1 L_2}{M_2}, \\
&K_2 = -\frac{\left[M_2^2 bL_2 M_1(\lambda_2^2 + 8\mu_2) - M_1^3 bL_2(\lambda_1^2 + 8\mu_1) + M_2(K_1 + 8bb_0 M_1)\right]}{M_1}, \\
&a_5 = b_5 = 0,
\end{aligned} \tag{35}$$

where $b_0, b, \lambda_1, \lambda_2, L_2, K_1, \mu_1, \mu_2, M_1$ and M_2 are arbitrary constants.

Substituting Equation (35) into Equations (33) and (34), we get:

$$u(\xi,\eta) = -\frac{\left[M_1^3 bL_2(\lambda_1^2+8\mu_1) - M_2(K_1+4bb_0M_1)\right]}{4bM_1L_2} - \frac{3M_1^2\lambda_1}{2}\left(\frac{G'(\xi)}{G(\xi)}\right) - \frac{3M_1^2}{2}\left(\frac{G'(\xi)}{G(\xi)}\right)^2 - \frac{3M_2^2\lambda_2}{2}\left(\frac{H'(\eta)}{H(\eta)}\right) - \frac{3M_2^2}{2}\left(\frac{H'(\eta)}{H(\eta)}\right)^2, \tag{36}$$

$$v(\xi,\eta) = b_0 + \frac{3M_1^2\lambda_1L_2}{2M_2}\left(\frac{G'(\xi)}{G(\xi)}\right) + \frac{3M_1^2L_2}{2M_2}\left(\frac{G'(\xi)}{G(\xi)}\right)^2 - \frac{3M_2\lambda_2L_2}{2}\left(\frac{H'(\eta)}{H(\eta)}\right) - \frac{3M_2L_2}{2}\left(\frac{H'(\eta)}{H(\eta)}\right)^2, \tag{37}$$

Where

$$\xi = M_1x + K_1t - \frac{M_1L_2}{M_2}y,$$
$$\eta = M_2x - \frac{\left[M_2^2bL_2M_1(\lambda_2^2+8\mu_2) - M_1^3bL_2(\lambda_1^2+8\mu_1) + M_2(K_1+8bb_0M_1)\right]}{M_1}t + L_2y. \tag{38}$$

We have the following families of exact solutions:

Family 1

If $\lambda_1^2 - 4\mu_1 > 0,$ and $\lambda_2^2 - 4\mu_2 > 0,$ we get the double soliton solutions of Equation (28) which have the following form:

$$u(\xi,\eta) = -\frac{\left[-M_1^3\lambda_1^2bL_2 + 16M_1^3bL_2\mu_1 - 2K_1M_2 - 8bM_2b_0M_1 - 3M_2^2\lambda_2^2L_2^2M_1b\right]}{8bM_1L_2} - \frac{3M_1^2(\lambda_1^2-4\mu_1)}{8}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}\right)^2 - \frac{3M_2^2(\lambda_2^2-4\mu_2)}{8}\left(\frac{C\cosh(\frac{1}{2}\sqrt{(\lambda_2^2-4\mu_2)}\,\eta) + D\sinh(\frac{1}{2}\sqrt{(\lambda_2^2-4\mu_2)}\,\eta)}{C\sinh(\frac{1}{2}\sqrt{(\lambda_2^2-4\mu_2)}\,\eta) + D\cosh(\frac{1}{2}\sqrt{(\lambda_2^2-4\mu_2)}\,\eta)}\right)^2,$$

$$v(\xi,\eta) = -\frac{\left[-8b_0M_2 + 3M_1^2\lambda_1^2L_2 + 3M_2^2\lambda_2^2L_2\right]}{8M_2} + \frac{3M_1^2L_2(\lambda_1^2-4\mu_1)}{8M_2}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}\right)^2 - \frac{3M_2L_2(\lambda_2^2-4\mu_2)}{8}\left(\frac{C\cosh(\frac{1}{2}\sqrt{(\lambda_2^2-4\mu_2)}\,\eta) + D\sinh(\frac{1}{2}\sqrt{(\lambda_2^2-4\mu_2)}\,\eta)}{C\sinh(\frac{1}{2}\sqrt{(\lambda_2^2-4\mu_2)}\,\eta) + D\cosh(\frac{1}{2}\sqrt{(\lambda_2^2-4\mu_2)}\,\eta)}\right)^2. \tag{39}$$

Family 2

If $\lambda_1^2 - 4\mu_1 > 0,$ and $\lambda_2^2 - 4\mu_2 < 0,$ we get the complexiton soliton solutions of Equation (28) which have the following form:

$$u(\xi,\eta) = -\frac{\left[-M_1^3\lambda_1^2 bL_2 + 16M_1^3 bL_2\mu_1 - 2K_1M_2 - 8bM_2b_0M_1 - 3M_2^2\lambda_2^2L_2^2M_1b\right]}{8bM_1L_2}$$
$$-\frac{3M_1^2(\lambda_1^2-4\mu_1)}{8}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}\right)^2$$
$$-\frac{3M_2^2(4\mu_2-\lambda_2^2)}{8}\left(\frac{-C\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}{C\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}\right)^2,$$
$$v(\xi,\eta) = -\frac{\left[-8b_0M_2 + 3M_1^2\lambda_1^2L_2 + 3M_2^2\lambda_2^2L_2\right]}{8M_2}$$
$$+\frac{3M_1^2L_2(\lambda_1^2-4\mu_1)}{8M_2}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}\right)^2$$
$$-\frac{3M_2L_2(4\mu_2-\lambda_2^2)}{8}\left(\frac{-C\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}{C\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}\right)^2. \tag{40}$$

Family 3

If $\lambda_1^2 - 4\mu_1 > 0,$ and $\lambda_2^2 - 4\mu_2 = 0,$ we get:

$$u(\xi,\eta) = -\frac{\left[-M_1^3\lambda_1^2 bL_2 + 16M_1^3 bL_2\mu_1 - 2K_1M_2 - 8bM_2b_0M_1 - 3M_2^2\lambda_2^2L_2^2M_1b\right]}{8bM_1L_2}$$
$$-\frac{3M_1^2(\lambda_1^2-4\mu_1)}{8}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}\right)^2$$
$$-\frac{3M_2^2}{2}\left(\frac{D}{C+D\eta}\right)^2,$$
$$v(\xi,\eta) = -\frac{\left[-8b_0M_2 + 3M_1^2\lambda_1^2L_2 + 3M_2^2\lambda_2^2L_2\right]}{8M_2} - \frac{3M_2L_2}{2}\left(\frac{D}{C+D\eta}\right)^2$$
$$+\frac{3M_1^2L_2(\lambda_1^2-4\mu_1)}{8M_2}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}\right)^2. \tag{41}$$

Family 4

If $\lambda_1^2 - 4\mu_1 < 0,$ and $\lambda_2^2 - 4\mu_2 < 0,$ we get the double triangular function solutions of equation (28) which have the

following form:

$$u(\xi,\eta)=-\frac{\left[-M_1^3\lambda_1^2bL_2+16M_1^3bL_2\mu_1-2K_1M_2-8bM_2b_0M_1-3M_2^2\lambda_2^2L_2^2M_1b\right]}{8bM_1L_2}$$
$$-\frac{3M_1^2(4\mu_1-\lambda_1^2)}{8}\left(\frac{-A\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}{A\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}\right)^2$$
$$-\frac{3M_2^2(4\mu_2-\lambda_2^2)}{8}\left(\frac{-C\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}{C\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}\right)^2,$$
$$v(\xi,\eta)=-\frac{\left[-8b_0M_2+3M_1^2\lambda_1^2L_2+3M_2^2\lambda_2^2L_2\right]}{8M_2}$$
$$+\frac{3M_1^2L_2(4\mu_1-\lambda_1^2)}{8M_2}\left(\frac{-A\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}{A\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}\right)^2$$
$$-\frac{3M_2L_2(4\mu_2-\lambda_2^2)}{8}\left(\frac{-C\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}{C\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}\right)^2. \tag{42}$$

Family 5

If $\lambda_1^2-4\mu_1<0,$ and $\lambda_2^2-4\mu_2=0,$ we get:

$$u(\xi,\eta)=-\frac{\left[-M_1^3\lambda_1^2bL_2+16M_1^3bL_2\mu_1-2K_1M_2-8bM_2b_0M_1-3M_2^2\lambda_2^2L_2^2M_1b\right]}{8bM_1L_2}$$
$$-\frac{3M_1^2(4\mu_1-\lambda_1^2)}{8}\left(\frac{-A\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}{A\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\xi)+B\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}\right)^2$$
$$-\frac{3M_2^2}{2}\left(\frac{D}{C+D\eta}\right)^2,$$
$$v(\xi,\eta)=-\frac{\left[-8b_0M_2+3M_1^2\lambda_1^2L_2+3M_2^2\lambda_2^2L_2\right]}{8M_2}-\frac{3M_2L_2}{2}\left(\frac{D}{C+D\eta}\right)^2$$
$$+\frac{3M_1^2L_2(4\mu_1-\lambda_1^2)}{8M_2}\left(\frac{-A\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}{A\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\xi)+B\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}\right)^2. \tag{43}$$

Family 6

If $\lambda_1^2-4\mu_1=0,$ and $\lambda_2^2-4\mu_2=0,$ we get:

$$u(\xi,\eta) = -\frac{\left[-M_1^3\lambda_1^2 bL_2 + 16M_1^3 bL_2\mu_1 - 2K_1M_2 - 8bM_2b_0M_1 - 3M_2^2\lambda_2^2L_2^2M_1b\right]}{8bM_1L_2}$$
$$-\frac{3M_1^2}{2}\left(\frac{B}{A+B\xi}\right)^2 - \frac{3M_2^2}{2}\left(\frac{D}{C+D\eta}\right)^2,$$
$$v(\xi,\eta) = -\frac{\left[-8b_0M_2 + 3M_1^2\lambda_1^2L_2 + 3M_2^2\lambda_2^2L_2\right]}{8M_2}$$
$$+\frac{3M_1^2L_2}{2M_2}\left(\frac{B}{A+B\xi}\right)^2 - \frac{3M_2L_2}{2}\left(\frac{D}{C+D\eta}\right)^2. \tag{44}$$

We should point out that not only the (39) to (44) are the solutions obtained in this example, but also we have some new solutions corresponding to double solitary like wave solutions, double trigonometric function solutions and complexiton soliton solutions of the (2+1)-dimensional breaking soliton equations, which are omitted here for simplicity .

Example 3: The (2+1)-dimensional Nizhnik-Novikov Veselov equations

Here, we study the (2 +1)-dimensional Nizhnik-Novikov Veselov equations in the following form (Zhang et al., 2008):

$$u_t - u_{xxx} - 3(uv)_x = 0,$$
$$u_y - v_y = 0. \tag{45}$$

We suppose the traveling wave transformations in the following form:

$$u(x,y,t) = u(\xi,\eta), \qquad v(x,y,t) = v(\xi,\eta), \tag{46}$$

Where

$$\xi = M_1x + K_1t + L_1y, \qquad \eta = M_2x + K_2t + L_2y, \tag{47}$$

and M_1, M_2, L_1, L_2, K_1 and K_2 are arbitrary constants to be determined later.

The solutions of Equation (45) can be expressed by a polynomial in (G'/G) and (H'/H) as the following form:

$$u(\xi,\eta) = a_0 + \sum_{k=1}^{N}\left(\sum_{i+j=k} a_i^j\left(\frac{G'(\xi)}{G(\xi)}\right)^i\left(\frac{H'(\eta)}{H(\eta)}\right)^j\right), \tag{48}$$

$$v(\xi,\eta) = b_0 + \sum_{k=1}^{M}\left(\sum_{i+j=k} b_i^j\left(\frac{G'(\xi)}{G(\xi)}\right)^i\left(\frac{H'(\eta)}{H(\eta)}\right)^j\right), \tag{49}$$

where $a_0, b_0, a_i^j, (i,j=1,...,N)$ and $b_i^j(i,j=1,...,M)$ are arbitrary constants, while $G(\xi)$ and $H(\eta)$ satisfy the second order linear ODE's (5). Considering the homogeneous balance between the highest-order derivatives and

nonlinear terms in Equation (45) we have $N = M = 2$, and consequently we can rewrite Equations (48) and (49) in the following form:

$$u(\xi,\eta) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 + a_3\left(\frac{H'(\eta)}{H(\eta)}\right) + a_4\left(\frac{H'(\eta)}{H(\eta)}\right)^2 + a_5\left(\frac{G'(\xi)}{G(\xi)}\right)\left(\frac{H'(\eta)}{H(\eta)}\right), \tag{50}$$

$$v(\xi,\eta) = b_0 + b_1\left(\frac{G'(\xi)}{G(\xi)}\right) + b_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 + b_3\left(\frac{H'(\eta)}{H(\eta)}\right) + b_4\left(\frac{H'(\eta)}{H(\eta)}\right)^2 + b_5\left(\frac{G'(\xi)}{G(\xi)}\right)\left(\frac{H'(\eta)}{H(\eta)}\right), \tag{51}$$

where $a_i, b_i (i = 0,1,...,5)$ are arbitrary constants to be determined. With the aid of Maple, we substitute (50), (51) along with (5) into Equation (45) and set the coefficients of $(G'/G)^i$ and $(H'/H)^i$ to be zero, yield a set of over determined algebraic equations with respect to $a_i, b_i (i = 0,1,...,5), \lambda_1, \lambda_2, \mu_1, \mu_2, M_1, M_2, L_1, L_2, K_1$ and K_2. On using the Maple software package, we solve the over determined algebraic equations, and consequently we get the following results:

$$\begin{aligned}
&b_0 = -\frac{\left[-L_2K_2 + 8M_2^3L_2\mu_2 + M_2^3L_2\lambda_2^2 + 3a_0M_2^2\right]}{3M_2L_2}, \\
&a_1 = \frac{2M_1^2L_2\lambda_1}{M_2}, \qquad a_2 = \frac{2M_1^2L_2}{M_2}, \qquad a_3 = -2M_2L_2\lambda_2, \\
&a_4 = -2M_2L_2, \qquad b_1 = -2M_1^2\lambda_1, \qquad b_2 = -2M_1^2, \\
&b_3 = -2\lambda_2M_2^2, \qquad b_4 = -2M_2^2, \qquad L_1 = -\frac{L_2M_1}{M_2}, \\
&K_1 = \frac{M_1(L_2M_1^2M_2\lambda_1^2 + 8L_2M_2M_1^2\mu_1 + L_2K_2 - 8M_2^3L_2\mu_2 - M_2^3L_2\lambda_2^2 - 6a_0M_2^2)}{L_2M_2}, \\
&a_5 = b_5 = 0,
\end{aligned} \tag{52}$$

where $a_0, L_2, K_2, \mu_1, \mu_2, \lambda_1, \lambda_2, M_1$ and M_2 are arbitrary constants.

Substituting Equation (52) into Equations (50) and (51), we get:

$$u(\xi,\eta) = a_0 + \frac{2M_1^2L_2\lambda_1}{M_2}\left(\frac{G'(\xi)}{G(\xi)}\right) + \frac{2M_1^2L_2}{M_2}\left(\frac{G'(\xi)}{G(\xi)}\right)^2 - 2M_2L_2\lambda_2\left(\frac{H'(\eta)}{H(\eta)}\right) - 2M_2L_2\left(\frac{H'(\eta)}{H(\eta)}\right)^2, \tag{53}$$

$$v(\xi,\eta)=-\frac{\left[-L_2K_2+8M_2^3L_2\mu_2+M_2^3L_2\lambda_2^2+3a_0M_2^2\right]}{3M_2L_2}-2M_1^2\lambda_1\left(\frac{G'(\xi)}{G(\xi)}\right)$$
$$-2M_1^2\left(\frac{G'(\xi)}{G(\xi)}\right)^2-2\lambda_2M_2^2\left(\frac{H'(\eta)}{H(\eta)}\right)-2M_2^2\left(\frac{H'(\eta)}{H(\eta)}\right)^2, \tag{54}$$

Where

$$\xi=M_1x+\frac{M_1(L_2M_1^2M_2\lambda_1^2+8L_2M_2M_1^2\mu_1+L_2K_2-8M_2^3L_2\mu_2-M_2^3L_2\lambda_2^2-6a_0M_2^2)}{L_2M_2}t-\frac{L_2M_1}{M_2}y,$$
$$\eta=M_2x+K_2t+L_2y. \tag{55}$$

We have the following families of exact solutions:

Family 1

If $\lambda_1^2-4\mu_1>0,$ and $\lambda_2^2-4\mu_2>0,$ we get the double soliton solutions of Equation (45) which have the following form:

$$u(\xi,\eta)=\frac{2M_2a_0-M_1^2L_2\lambda_1^2+M_2^2L_2\lambda_2^2}{2M_2}$$
$$+\frac{2M_1^2L_2}{M_2}\left(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}\right)^2$$
$$-2M_2L_2\left(\frac{1}{2}\sqrt{(\lambda_2^2-4\mu_2)}\,\frac{C\cosh(\frac{1}{2}\sqrt{(\lambda_2^2-4\mu_2)}\,\eta)+D\sinh(\frac{1}{2}\sqrt{(\lambda_2^2-4\mu_2)}\,\eta)}{C\sinh(\frac{1}{2}\sqrt{(\lambda_2^2-4\mu_2)}\,\eta)+D\cosh(\frac{1}{2}\sqrt{(\lambda_2^2-4\mu_2)}\,\eta)}\right)^2,$$
$$v(\xi,\eta)=\frac{2L_2K_2-16M_2^3L_2\mu_2+M_2^3L_2\lambda_2^2-6a_0M_2^2+3L_2M_1^2M_2\lambda_1^2}{6M_2L_2}$$
$$-2M_1^2\left(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}\right)^2$$
$$-2M_2^2\left(\frac{1}{2}\sqrt{(\lambda_2^2-4\mu_2)}\,\frac{C\cosh(\frac{1}{2}\sqrt{(\lambda_2^2-4\mu_2)}\,\eta)+D\sinh(\frac{1}{2}\sqrt{(\lambda_2^2-4\mu_2)}\,\eta)}{C\sinh(\frac{1}{2}\sqrt{(\lambda_2^2-4\mu_2)}\,\eta)+D\cosh(\frac{1}{2}\sqrt{(\lambda_2^2-4\mu_2)}\,\eta)}\right)^2. \tag{56}$$

Family 2

If $\lambda_1^2-4\mu_1>0,$ and $\lambda_2^2-4\mu_2<0,$ we get the complexiton soliton solutions of Equation (45) which have the following form:

$$u(\xi,\eta)=\frac{2M_2a_0-M_1^2L_2\lambda_1^2+M_2^2L_2\lambda_2^2}{2M_2}$$
$$+\frac{2M_1^2L_2}{M_2}\left(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}\right)^2$$
$$-2M_2L_2\left(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\frac{-C\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}{C\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}\right)^2,$$
$$v(\xi,\eta)=\frac{2L_2K_2-16M_2^3L_2\mu_2+M_2^3L_2\lambda_2^2-6a_0M_2^2+3L_2M_1^2M_2\lambda_1^2}{6M_2L_2}$$
$$-2M_1^2\left(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}\right)^2$$
$$-2M_2^2\left(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\frac{-C\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}{C\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}\right)^2. \tag{57}$$

Family 3

If $\lambda_1^2-4\mu_1>0,$ and $\lambda_2^2-4\mu_2=0,$ we get:

$$u(\xi,\eta)=\frac{2M_2a_0-M_1^2L_2\lambda_1^2+M_2^2L_2\lambda_2^2}{2M_2}-2M_2L_2\left(\frac{D}{C+D\eta}\right)^2$$
$$+\frac{2M_1^2L_2}{M_2}\left(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}\right)^2,$$
$$v(\xi,\eta)=\frac{2L_2K_2-16M_2^3L_2\mu_2+M_2^3L_2\lambda_2^2-6a_0M_2^2+3L_2M_1^2M_2\lambda_1^2}{6M_2L_2}-2M_2^2\left(\frac{D}{C+D\eta}\right)^2$$
$$-2M_1^2\left(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)+B\cosh(\frac{1}{2}\sqrt{(\lambda_1^2-4\mu_1)}\,\xi)}\right)^2. \tag{58}$$

Family 4

If $\lambda_1^2-4\mu_1<0,$ and $\lambda_2^2-4\mu_2<0,$ we get the double triangular function solutions of Equation (45) which have the following form:

$$u(\xi,\eta)=\frac{2M_2a_0-M_1^2L_2\lambda_1^2+M_2^2L_2\lambda_2^2}{2M_2}$$
$$+\frac{2M_1^2L_2}{M_2}\left(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\frac{-A\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}{A\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}\right)^2$$
$$-2M_2L_2\left(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\frac{-C\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}{C\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}\right)^2,$$
$$v(\xi,\eta)=\frac{2L_2K_2-16M_2^3L_2\mu_2+M_2^3L_2\lambda_2^2-6a_0M_2^2+3L_2M_1^2M_2\lambda_1^2}{6M_2L_2}$$
$$-2M_1^2\left(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\frac{-A\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}{A\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}\right)^2$$
$$-2M_2^2\left(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\frac{-C\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}{C\cos(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)+D\sin(\frac{1}{2}\sqrt{(4\mu_2-\lambda_2^2)}\,\eta)}\right)^2. \tag{59}$$

Family 5

If $\lambda_1^2-4\mu_1<0,$ and $\lambda_2^2-4\mu_2=0,$ we get:

$$u(\xi,\eta)=\frac{2M_2a_0-M_1^2L_2\lambda_1^2+M_2^2L_2\lambda_2^2}{2M_2}-2M_2L_2\left(\frac{D}{C+D\eta}\right)^2$$
$$+\frac{2M_1^2L_2}{M_2}\left(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\frac{-A\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}{A\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}\right)^2,$$
$$v(\xi,\eta)=\frac{2L_2K_2-16M_2^3L_2\mu_2+M_2^3L_2\lambda_2^2-6a_0M_2^2+3L_2M_1^2M_2\lambda_1^2}{6M_2L_2}-2M_2^2\left(\frac{D}{C+D\eta}\right)^2$$
$$-2M_1^2\left(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\frac{-A\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}{A\cos(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)+B\sin(\frac{1}{2}\sqrt{(4\mu_1-\lambda_1^2)}\,\xi)}\right)^2. \tag{60}$$

Family 6

If $\lambda_1^2-4\mu_1=0,$ and $\lambda_2^2-4\mu_2=0,$ we get:

$$u(\xi,\eta)=\frac{2M_2a_0-M_1^2L_2\lambda_1^2+M_2^2L_2\lambda_2^2}{2M_2}$$
$$+\frac{2M_1^2L_2}{M_2}\left(\frac{B}{A+B\xi}\right)^2-2M_2L_2\left(\frac{D}{C+D\eta}\right)^2,$$
$$v(\xi,\eta)=\frac{2L_2K_2-16M_2^3L_2\mu_2+M_2^3L_2\lambda_2^2-6a_0M_2^2+3L_2M_1^2M_2\lambda_1^2}{6M_2L_2}$$
$$-2M_1^2\left(\frac{B}{A+B\xi}\right)^2-2M_2^2\left(\frac{D}{C+D\eta}\right)^2. \quad (61)$$

We should point out that not only the 56 to 61 are the solutions obtained in this example, but also we have some new solutions corresponding to double solitary like wave solutions, double trigonometric function solutions and complexiton soliton solutions of the (2+1)-dimensional Nizhnik-Novikov Veselov equations, which are omitted here for simplicity .

CONCLUSION

In this paper, we have used the double soliton solutions method to obtain many new types of the exact solutions for some nonlinear partial differential equations in the mathematical physics. This method is concise and effective, and it can be applied to other nonlinear evolution equations in mathematical physics.

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