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Short Communication

A new quaternion: Unit vector division quaternion

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It is well known that, quaternion algebra allows the defining division of 3-D vectors. Division of vectors has many applications in applied mathematics and mechanics. However, indicial notation is widely used in the branches mentioned above. It is shown in this paper that division of orthonormal vectors can be conveniently handled by a simple operator. This operator can be used in the division of two arbitrary vectors and allows the expression of vector division using indicial notation. This new operator, which is a quaternion itself, can be also used in expressing quaternion product of two vectors, and the derivative of a vector with respect to a vector.

Key words: Quaternion, vector quaternions, Einstein's summation convention

INTRODUCTION

A quaternion is defined by the combination of four real numbers:

$$\mathbb{A} = (a_0, a_1, a_2, a_3) \tag{1}$$

Where a_o, a_1, a_2, a_3, are real numbers and A is a quaternion. Castellar characters denote quaternions. It is possible to consider a quaternion as a combination of a scalar and a vector. A quaternion A = (a_o, a_1, a_2, a_3) can be expressed in the form of (Kosenko 1998):

$$\mathbb{A} = (a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \tag{2}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the orthogonal unit vectors in a Cartesian coordinate system. Boldface letters denote vectors. Equation (2) can be written as:

$$\mathbb{A} = (\alpha + \mathbf{u}) \tag{3}$$

Where α is a scalar and **u** is a vector in three dimensional Euclidean space. Any vector can be interpreted as a vector quaternion whose scalar part is zero:

$$\mathbb{A} = (0 + \mathbf{u}) \tag{4}$$

The product of two quaternions of A = (α + **u**) and B = (β + **v**), (where β is a scalar and **v** is a vector) is:

$$\mathbb{AB} = (\alpha+\mathbf{u})(\beta+\mathbf{v}) = (\alpha\beta - \mathbf{u}\cdot\mathbf{v} + \alpha\mathbf{v} + \beta\mathbf{u} + \mathbf{u}\times\mathbf{v}) \tag{5}$$

where ($\cdot$) stands for the dot product and (x) stands for the cross product defined for vectors. Division of two quaternions can be expressed in two types; left division and right division, respectively:

$$\left.\frac{\mathbb{A}}{\mathbb{B}}\right|_L = \mathbb{B}^{-1}\mathbb{A} \quad \text{and} \quad \left.\frac{\mathbb{A}}{\mathbb{B}}\right|_R = \mathbb{A}\,\mathbb{B}^{-1} \tag{6}$$

The difference in left division and right division comes from the fact that the quaternion product is not commutative.

The inverse of a quaternion given in equation (6) can be obtained as:

$$\mathbb{A}^{-1} = \frac{\mathbb{A}^*}{\|\mathbb{A}\|} \tag{7}$$

where $\mathbb{A}^*$ is the conjugate and $\|\mathbb{A}\|$ is the norm of the mentioned quaternion. They are defined as:

$$\mathbb{A}^* = (\alpha - \mathbf{u}) \text{ and } \|\mathbb{A}\| = a_0^2 + a_1^2 + a_2^2 + a_3^2 \quad (8)$$

UNIT VECTOR DIVISION QUATERNION

Vectors can be divided as well, since they can be considered as vector quaternions. The only problem is to choose the type of the division, to be left or right. This will change only the cross product term in quaternion product given in equation (5), which is only a sign convention problem. As a result, division of two orthonormal vectors can be obtained:

$$\left.\frac{\mathbf{e}_i}{\mathbf{e}_j}\right|_L = \mathbb{Q}_{ij} \quad \text{and} \quad \left.\frac{\mathbf{e}_i}{\mathbf{e}_j}\right|_R = \mathbb{Q}_{ji} \quad (9)$$

where Q_{ij} is the unit vector division quaternion, and can be defined as:

$$Q_{ij} = \begin{cases} (1+\mathbf{0}) & \text{if } i = j \\ (0+\mathbf{e}_i \times \mathbf{e}_j) & \text{if } i \neq j \end{cases} \quad (10)$$

To the best of the knowledge of the author, it is the first time that the quantity defined above is presented in this paper. In this text, left division will be preferred.

Although the above quantity is a quaternion, it can not be directly applied to quaternions that are in form of A = (α+ u). Since Einstein's summation convention need to be kept valid (Goldstein, 1980) one can not say or write an expression in the form of Q_{ij} (α + u). Because, if the vector u in the quaternion is expressed in terms of unit vectors with one of the indices in unit vector division quaternion, like in the form of u = $u_i e_i$, an index would be repeated more than once, which is not allowed. If another index is used, this time, the decision in Equation (10) can not be made. Since unit vector division quaternion is not a tensor, it can not be said whether it is symmetric or antisymmetric.

APPLICATIONS

Unit vector division quaternion gives the general product of two vectors given by Gibbs as stated by Gurmen (1985):

$$(\mathbf{u})(\mathbf{v}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\times\mathbf{v} = \mathbb{Q}_{ij}\, u_i v_j \quad (11)$$

It should not be forgotten that the repeated indices in Equation (11) should be summed up from one to three. If the term in the right-hand side of equation (11) is opened, it is obtained as:

$$\begin{aligned} Q_{ij}u_i v_j &= Q_{11}u_1v_1 + Q_{12}u_1v_2 + Q_{13}u_1v_{31} \\ &+ Q_{21}u_2v_1 + Q_{22}u_2v_2 + Q_{23}u_2v_3 \\ &+ Q_{31}u_3v_1 + Q_{32}u_3v_2 + Q_{33}u_3v_3 \end{aligned} \quad (12)$$

If the definition given in Equation (10) is substituted in equation (12), it is respectively obtained as:

$$\begin{aligned} Q_{ij}u_i v_j &= u_1v_1 + u_2v_2 + u_3v_3 \\ &+ (u_2v_3 - u_3v_2)\mathbf{e}_1 + (u_3v_1 - u_1v_3)\mathbf{e}_2 + (u_1v_2 - u_2v_1)\mathbf{e}_3 \\ &= \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\times\mathbf{v} \end{aligned} \quad (13)$$

Then, with the same token, the product of two vector quaternions given in Equation (5) can be expressed with the help of unit vector division quaternion:

$$(0+\mathbf{u})\,(0+\mathbf{v}) = -\mathrm{Q}_{ji}\, u_i\, v_j = -\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\times\mathbf{v} \quad (14)$$

Unit vector division quaternion can not be used more than once with a common index when they are used in product terms; since the product of this quaternion by two vectors can give a quaternion. Thus, an expression in the form of $Q_{ij}\, Q_{jk}\, u_i\, v_k$ is meaningless even if it obeys the rules of indicial notation. When this expression is opened according to Einstein's summation convention, it is obvious that there will be a term $Q_{12}\, Q_{23}\, u_1\, v_3$ if i = 1, j = 2, k = 3. It is known that Q_{12} = $\mathbf{e}_3$ and Q_{23} = $\mathbf{e}_1$. Since the type of product of these two vectors are not defined in this operation, using unit vector division quaternion more than once with a common index should not be allowed.

Unit vector division quaternion can be used to obtain the division of two vectors in arbitrary directions, if they are considered as two vector quaternions:

$$\frac{\mathbf{u}}{\mathbf{v}} = \frac{(0+\mathbf{u})}{(0+\mathbf{v})} = \frac{1}{v^2}\,\mathbb{Q}_{ji}\, v_i\, u_j \quad (15)$$

However, it should be noted that the vector division defined in Equation (15) is neither the inverse of cross product nor the inverse of dot product. It can only for the vectors that are divided according to the physical definition of the problem.

Last, the derivative of a vector with respect to a vector can be defined:

$$\frac{d\mathbf{u}}{d\mathbf{v}} = \frac{1}{(dv)^2}\,\mathrm{Q}_{ji}\, dv_i\, du_j \quad (16)$$

It should be kept in mind that the right-hand side of equation (16) contains vectors as well as scalars. So it does not have to be invariant under the axis transformations.

CONCLUSIONS

In this paper, a new quaternion is defined and some applications of this new quaternion are given. Quaternion product and division of two vectors can be obtained in indicial notation with the help of this new quaternion. Division of two vectors and derivative of a vector, with respect to a vector, can be applied to many branches of applied mathematics and mechanics.

REFERENCES

Goldstein H (1980). Classical Mechanics, Addison Wesley, Massachusetts, pp. 132-133.

Gurmen H (1985). A Proposal for Type Identification Analysis of Physical Quantities, Int. J. Elect., 59: 495-499.

Kosenko I (1998). Integration of the equations of a rotational motion of a rigid body in quaternion algebra. The Euler case, J. Appl. Math. Mech., 62: 193-200.