

Full Length Research Paper

Functional projective lag synchronization of chaotic systems with disturbances

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This paper considers the synchronization problem of chaotic systems. Functional projective lag synchronization (FPLS), which is a generalized synchronization concept recently developed, was investigated. Based on Lyapunov stability theory, a novel stability criterion for the synchronization between master and slave chaotic systems was derived. The proposed method was applied to unified chaotic systems in order to show the effectiveness of our results.

Key words: Chaotic system, functional projective synchronization, lag synchronization, lyapunov method.

INTRODUCTION

Since Lorenz (1963) has discovered chaotic phenomenon in meteorology, various chaotic systems have introduced and studied (Chen et al., 1999; Lü et al., 2002; Rössler, 1976). Chaotic system is a sort of nonlinear systems that has unpredictable behaviour and sensitivity to initial conditions and parameter uncertainties. Due to these properties, synchronization problems easily found in many physical and biological systems such as heart beat, walking, coordinated robot motion and so on. The synchronization between chaotic systems is a more interesting issue, because chaotic systems are hard to expect their behaviour and very sensitive to initial conditions. Since, Pecora and Carroll's (1990) method for synchronizing two identical chaotic systems, the various control schemes for the problem such as back-stepping design (Park, 2006; Vincent et al., 2007), static error feedback approach (Vincent, 2005; Li et al., 2005), observer-based control (Bowong, 2008), time-delay feedback approach (Park et al., 2008¹), parameters adaptive control (Yassen, 2006; Park et al., 2009; Lee et al., 2009), and dynamic feedback approach (Park et al., 2007; Park et al., 2008²) have been proposed.

Originally, chaos synchronization refers to the state in which the master (or drive) and the slave (or response) systems have precisely identical trajectories for time to

infinity. We usually regard such synchronization as complete synchronization or identical synchronization. During the last decade, various synchronization methods such as anti-synchronization (Song et al., 2007), phase synchronization (Rosenblum et al., 1996), projective synchronization (PS) (Mainieri et al., 1999), generalized synchronization (Park, 2007), lag synchronization (Yu et al., 2007; Li, 2009), and functional projective synchronization (Du et al., 2008; Runzi, 2008) have been investigated. Note that functional projective synchronization is the state of the art and generalized concept of synchronization schemes. In this scheme, the error signal between master and slave systems can be synchronized up to a scaling function, but not as constant.

On the other hand, time delays are ubiquitous between master and slave communication in real implementation. Thus, it is natural to consider time delay when we deal with synchronization problem. In this regard, there are strong needs to considered the functional projective lag synchronization (FPLS). In this paper, we consider a generalized synchronization problem for chaotic systems with disturbances. Note that, the disturbance was not considered in most of existing work when the synchronization problem was tackled. Based on Lyapunov method and linear matrix inequality (LMI) framework, an existence criterion of stabilizing controllers for FPLS of the systems was presented.

This paper was organized as follows. In Section 2, the problem statement and master-slave synchronization

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scheme was presented. In section 3, a numerical example was given to demonstrate the effectiveness of the proposed idea. Finally, a conclusion was given in final section.

MAIN RESULTS

Consider the following master (drive) and slave (response) chaotic systems:

$$\dot{x}(t) = Ax(t) + f(x) + d_1(t), \tag{1}$$

$$\dot{y}(t) = Ay(t) + f(y) + d_2(t) + u(t), \tag{2}$$

where, $x(t) = (x_1, x_2, \dots, x_n)^T \in \mathfrak{R}^n$ and $y(t) = (y_1, y_2, \dots, y_n)^T \in \mathfrak{R}^n$ are state vectors of master and slave systems, respectively, $f(x): \mathfrak{R}^n \rightarrow \mathfrak{R}^n$ is a continuous nonlinear vector function, $A \in \mathfrak{R}^{n \times n}$ is a constant matrix, $d_1(t)$ and $d_2(t)$ are the disturbance signals bounded in magnitude $\|d_1(t)\| \leq \bar{d}_1$, $\|d_2(t)\| \leq \bar{d}_2$ and $u(t) = (u_1, u_2, \dots, u_n)^T \in \mathfrak{R}^n$ is the control input for synchronization between master system (1) and slave system (2). Let us define the error vector as

$$e(t) = y(t) - \alpha(t)x(t - \tau) \tag{3}$$

where, $\alpha(t)$ is a continuously differentiable bounded function satisfying $\max|\alpha(t)| = h$, τ is positive real time delay and $e(t) = (e_1, e_2, \dots, e_n)^T \in \mathfrak{R}^n$ with $e_i(t) = y_i(t) - \alpha(t)x_i(t - \tau)$.

Here, we give the definition of functional projective lag synchronization (FPLS) and a related remark.

Definition 1. It was said that, FPLS occurs between master system (1) and response to system (2) if the condition $\lim_{t \rightarrow \infty} \|y(t) - \alpha(t)x(t - \tau)\| = 0$ holds for a given scaling function $\alpha(t)$.

Remark 1. Chaos synchronization schemes such as complete synchronization, lag synchronization, anti-synchronization, and projective synchronization are special case of FPLS. When, $\alpha = 1; \tau = 0$, $\alpha = 1; \tau =$ positive constant, $\alpha = -1; \tau = 0$, and $\alpha =$ constant; $\tau = 0$, then FPLS becomes complete synchronization, lag synchronization, anti-synchronization and projective synchronization, respectively.

From the definition of error signal (3), then error dynamics is

$$\begin{aligned} \dot{e}(t) &= Ae(t) + f(y) - \alpha(t)f(x(t - \tau)) - \dot{\alpha}(t)x(t - \tau) \\ &+ d_2(t) - \alpha(t)d_1(t - \tau) + u(t) \end{aligned} \tag{4}$$

Now, we have the following main result.

Theorem 1. For a given positive constant ϵ , if there exists in a positive-definite matrix P and a matrix Y satisfying the following LMI

$$A^T P + PA + Y^T + Y + 2\epsilon I < 0, \tag{5}$$

Then, master system (1) and slave system (2) can be synchronized up to a scaling function $\alpha(t)$ via the controller:

$$\begin{aligned} u(t) &= Ke(t) + \dot{\alpha}(t)x(t - \tau) - f(y) + \alpha(t)f(x(t - \tau)) \\ &- \frac{d^2}{d\|e^T(t)P\| + \epsilon\|e(t)\|^2} Pe(t) \end{aligned} \tag{6}$$

where, K and P are control parameters which are determined later, and d is a constant satisfying $d_2(t) - \alpha(t)d_1(t - \tau) \leq \bar{d}_2 + h\bar{d}_1 \leq d$.

Proof. Substituting the control input (6) into Equation (4) gives

$$\dot{e}(t) = (A + K)e(t) + d_2(t) - \alpha(t)d_1(t - \tau) - \frac{d^2}{d\|e^T(t)P\| + \epsilon\|e(t)\|^2} Pe(t) \tag{7}$$

For stability analysis, let us consider the following Lyapunov function

$$V = \frac{1}{2} e^T(t) P e(t) \tag{8}$$

where, P is the positive definite matrix defined in (5).

The time derivative along the solution of system (7) is as follows:

$$\begin{aligned} \dot{V} &= e^T(t)(A^T P + PA + K^T P + PK)e(t) + 2e^T(t)P(d_2(t) - \alpha(t)d_1(t - \tau)) \\ &- 2e^T(t)P \frac{d^2}{d\|e^T(t)P\| + \epsilon\|e(t)\|^2} Pe(t) \\ &\leq e^T(t)(A^T P + PA + K^T P + PK)e(t) + 2\|e^T(t)P\|d - 2 \frac{\|e^T(t)P\|^2 d^2}{d\|e^T(t)P\| + \epsilon\|e(t)\|^2} \\ &= e^T(t)(A^T P + PA + K^T P + PK)e(t) + 2 \frac{d\|e^T(t)P\| \cdot \epsilon\|e(t)\|^2}{d\|e^T(t)P\| + \epsilon\|e(t)\|^2} \\ &\leq e^T(t)(A^T P + PA + K^T P + PK)e(t) + 2\epsilon\|e(t)\|^2 \\ &= e^T(t)(A^T P + PA + K^T P + PK + 2\epsilon I)e(t) \equiv -e^T(t)Qe(t) \end{aligned} \tag{9}$$

where, the well-known inequality $0 \leq ab/(a+b) \leq a \forall a, b > 0$ is used and $Q = -(A^T P + PA + K^T P + PK + 2\epsilon I)$. Obviously, the error

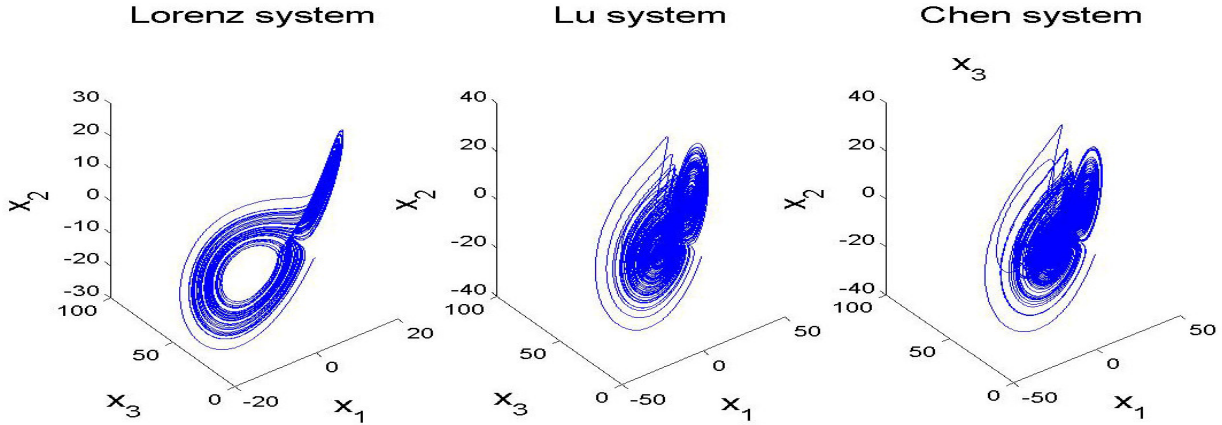


Figure 1. Chaotic motion of unified system.

system (7) is asymptotically stable if we can choose the matrix P , which makes matrix Q be positive definite. Finally, by defining $Y = PK$, the negativeness of Q is equivalent to the LMI given in Equation (5). Hence, if the LMI (5) holds, then one can conclude that the error system (4) is asymptotically stable. This completes the proof.

Remark 2. In order to solve the LMI (5) given in Theorem 1, Matlab's LMI Control Toolbox can be utilized, which implements state-of-the-art interior-point algorithms, which is significantly faster than classical convex optimization algorithms (Boyd et al., 1994). The feedback gain matrix K can be calculated from the $K = P^{-1}Y$ after finding the LMI solutions, P and Y from (5).

NUMERICAL SIMULATION

In this section, a numerical example was presented to show the effectiveness of our synchronization scheme. Consider the following unified chaotic master and slave systems (Lü et al., 2002):

Master system:

$$\begin{aligned} \dot{x}_1(t) &= (25s + 10)(x_2(t) - x_1(t)) + d_{11}(t) \\ \dot{x}_2(t) &= (28 - 35s)x_1(t) - x_1(t)x_3(t) \\ &\quad + (29s - 1)x_2(t) + d_{12}(t) \\ \dot{x}_3(t) &= x_1(t)x_2(t) - \frac{s+8}{3}x_3(t) + d_{13}(t) \end{aligned}$$

Slave system:

$$\begin{aligned} \dot{y}_1(t) &= (25s + 10)(y_2(t) - y_1(t)) + d_{21}(t) + u_1(t) \\ \dot{y}_2(t) &= (28 - 35s)y_1(t) - y_1(t)y_3(t) \\ &\quad + (29s - 1)y_2(t) + d_{22}(t) + u_2(t) \\ \dot{y}_3(t) &= y_1(t)y_2(t) - \frac{s+8}{3}y_3(t) + d_{23}(t) + u_3(t) \end{aligned} \quad (10)$$

It is well-known that the unified chaotic system (10) has chaotic behaviour for any $s \in [0,1]$. It is called the general Lorenz, Lü, and Chen system when $s \in [0,0.8)$, $s = 0.8$, and $s \in (0.8,1]$, respectively. In order to see chaotic motion of the system (10), let us take the initial condition $x(0) = (0,1,1)^T$. Figure 1 shows three chaotic motions of unified system at $s = 0, s = 0.8$ and $s = 1$, respectively.

Now, we can rewrite the system (10) in the form of Equation (1):

$$\begin{aligned} A &= \begin{bmatrix} -(25s+10) & 25s+10 & 0 \\ 28-35s & 29s-1 & 0 \\ 0 & 0 & -\frac{s+8}{3} \end{bmatrix}, \quad f(x) = \begin{bmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{bmatrix}, \quad f(y) = \begin{bmatrix} 0 \\ -y_1y_3 \\ y_1y_2 \end{bmatrix}, \\ d_1(t) &= \begin{bmatrix} d_{11}(t) \\ d_{12}(t) \\ d_{13}(t) \end{bmatrix}, \quad d_2(t) = \begin{bmatrix} d_{21}(t) \\ d_{22}(t) \\ d_{23}(t) \end{bmatrix}. \end{aligned} \quad (11)$$

In order to make functional projective lag synchronization of system (1) and (2) with disturbances, the scaling function is chosen as $\alpha(t) = 1.5 + \sin 2\pi t$, and the constant ε, h, τ and d are selected as 0.0001, 3, 1.5 and 2, respectively.

In the numerical simulation, the fourth-order Runge-Kutta method with sampling time 0.0001[sec] is used to solve differential equations in the paper. Initial conditions of master and slave systems are chosen as $x(0) = (-5, -7, -10)^T$ and $y(0) = (0, 2, -1)^T$, respectively. External disturbances are selected by satisfying that : $d_{11}(t) = \text{random number of } |d_{11}(t)| \leq 0.1$, $d_{12}(t) = \text{random number of } |d_{12}(t)| \leq 0.2$, $d_{13}(t) = 0$, $d_{21}(t) = \text{random number of } |d_{21}(t)| \leq 0.2$, $d_{22}(t) = 0$, $d_{23}(t) = \text{random number of } |d_{23}(t)| \leq 0.1$.

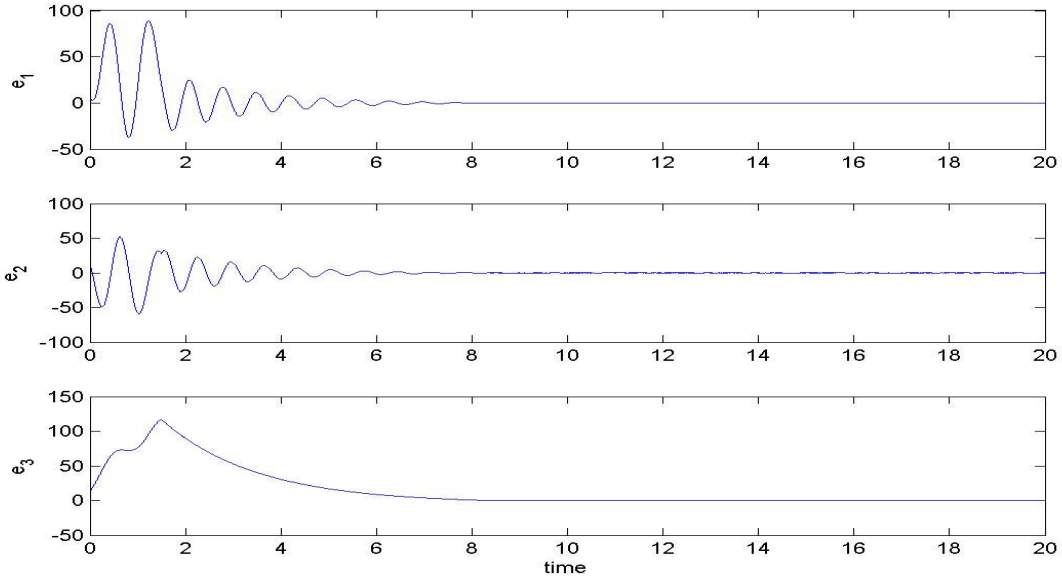


Figure 2. Error signals for case 1.

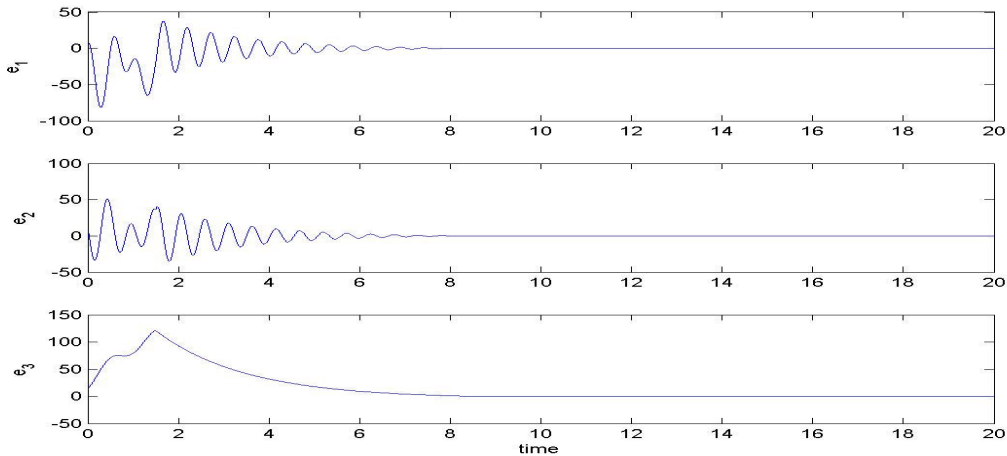


Figure 3. Error signals for case 2.

Here, in order to solve the LMI given in Theorem 1, let us utilize MATLAB's LMI Control Toolbox (Boyd et al., 1994), then, one can see that the LMI given in Eq. (9) is feasible and get possible solution sets for three cases as:

Case 1: Lorenz system: $s = 0$

$$P = 10^7 \times \begin{bmatrix} 3.4549 & 0 & 0 \\ 0 & 3.4549 & 0 \\ 0 & 0 & 3.4549 \end{bmatrix}, K = \begin{bmatrix} 95 & -19 & 0 \\ -19 & 0.5 & 0 \\ 0 & 0 & 21667 \end{bmatrix}$$

Case 2: Lü system: $s = 0.8$

$$P = 10^7 \times \begin{bmatrix} 2.3046 & 0 & 0 \\ 0 & 2.3046 & 0 \\ 0 & 0 & 2.3046 \end{bmatrix}, K = \begin{bmatrix} 29.5 & -15 & 0 \\ -15 & -22.7 & 0 \\ 0 & 0 & 2.4333 \end{bmatrix}$$

Case 3: Chen system: $s = 1$

$$P = 10^7 \times \begin{bmatrix} 2.0189 & 0 & 0 \\ 0 & 2.0189 & 0 \\ 0 & 0 & 2.0189 \end{bmatrix}, K = \begin{bmatrix} 34.5 & -14 & 0 \\ -14 & -28.5 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$$

Figures 2, 3 and 4 shows that, error signals of case 1, 2 and 3, respectively, go to zero as time goes infinity. This implies that, the FPLS of the system (10) was achieved

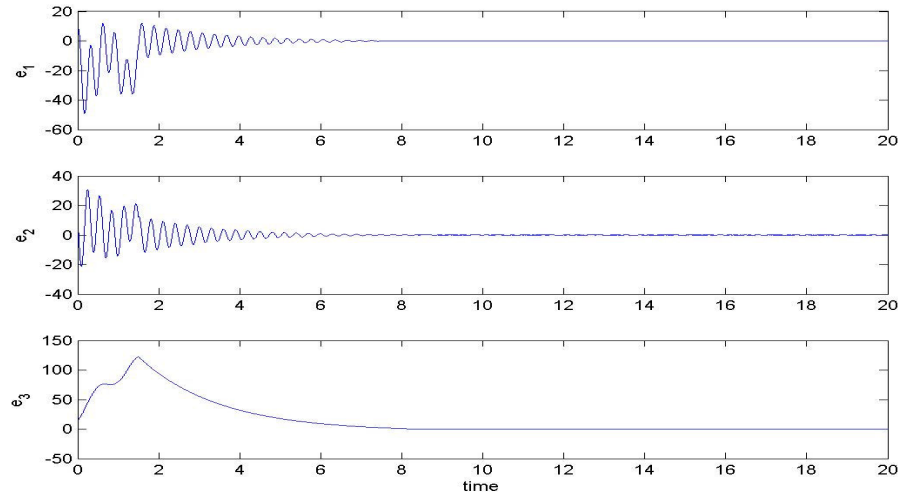


Figure 4. Error signals for case 3.

via our proposed control scheme.

CONCLUSION

In this paper, we have investigated the functional projective lag synchronization problem for a general class of unified chaotic systems. Based on Lyapunov method and LMI framework, the controller for our synchronization problem has designed to guarantee asymptotic stability for error dynamics. Numerical simulations show that our method is effective for FPLS. Finally, note that our control scheme was applied to various types of chaotic systems.

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