Scientific Research and Essays Vol. 6(14), pp. 2988-2997, 18 July, 2011
Available online at http://www.academicjournals.org/SRE
DOI: 10.5897/SRE11.246
ISSN 1992-2248 ©2011 Academic Journals

Full Length Research Paper

Diffusion in graded materials by decomposition method

Ali Sahin* and Ibrahim Karatay

Department of Mathematics, Fatih University, 34500 Buyukcekmece, Istanbul, Turkey.

Accepted 24 May, 2011

In this study, diffusion equation for composite materials was examined using a well-known Adomian Decomposition Method (ADM). Defining variable conductivity and heat capacity as an exponential function or a power function that represents Functionally Graded Materials (FGMs), one-dimensional diffusion equation with non-homogeneous boundary conditions was examined. First, using standard superposition method the diffusion equation is turned into non-homogeneous one with homogeneous boundary conditions. Then, using generalized Fourier series expansion, the resultant PDE is solved by using ADM. The results are compared with the solution obtained by eigenfunction expansion method.

Key words: Heat conduction, adomian decomposition method, functionally graded materials, eigenfunction expansion.

INTRODUCTION

In recent years, the Adomian decomposition method (ADM) has been applied to solve a wide class of deterministic and stochastic PDEs in Science and Engineering (Adomian, 1988; 1989; 1994; Cherruault, 1989). The method is modified by researchers to solve many problems whose models involve, linear or non-linear, differential equations, integral equations, partial differential equations and their systems (Wazwaz, 1999a; 2001). The advantage of the method is that it converges rapidly to a convergent series solution for linear as well as non-linear deterministic and stochastic equations. There is also no need for linearization and perturbation in this method. Because of these properties of the method, many different types of special problems such as Korteweg–de Vries (KdV) equation (Kaya et al., 2003a,b), Shivansky Equation (Momani et al., 2005), Emden-Fowler type equations (Aslanov, 2009) easily solved by ADM.

A powerful modification of ADM was proposed by Wazwaz (1999b) to accelerate the rapid convergence of the series solution. Using the modified technique, it may be obtained the exact solution for nonlinear equations without any need of Adomian polynomials. By modified ADM, the size of calculation is minimized in terms of the standard ADM.

*Corresponding author. E-mail: asahin@fatih.edu.tr.

Many researchers due to its importance in science and engineering applications have investigated the solution of heat conduction problem. Many attempts were made to solve heat conduction problem in homogeneous and inhomogeneous materials using analytical and numerical techniques. Jang (2007) examined solution of one dimensional nonhomogeneous parabolic type equation with variable coefficient using ADM. Gorguis and Benny Chan (2008) were compared the results of the solution of heat equation by using ADM and the traditional separation of variables method. In many problems, after calculation of heat conduction by ADM, results are controlled with either analytical or numerical solutions examined in the previous studies. Under periodic temperature conditions, the convergence of ADM in one dimensional heat equation was investigated by Lesnic (2002) using differential iteration method. It was resulted that ADM established better rates of convergence compare with the differential iteration method. Marwat and Asghar (2008), as two-step Adomian decomposition method, for a diffusion equation, modified the method of Adomian decomposition. In the application of the modified method, they showed that the generalized Fourier series instead of the trigonometric Fourier series is required to build up the solution.

In early 1980's, the concept of Functionally Graded Materials (FGMs) was proposed as an alternative to conventional thermal barrier ceramic coatings. FGMs are essentially two-phase particulate composites synthesized in such a way that the volume fractions of the constituents vary continuously in the thickness direction to give a predetermined composition profile. The composition profile, which varies from 0% ceramic at the interface to 100% ceramic near the surface, in turn, is selected in such a way that the resulting nonhomogeneous material exhibits the desired thermomechanical properties. The concept of FGMs could provide great flexibility in material design by controlling both the composition profile and the microstructure. Current and potential applications of the concept of FGMs include not only thermal barrier coatings of high temperature components but also wear-resistant coatings on load transfer components, armors or shields with improved impact resistance, and thermoelectric cells (Kaysser, 1999; Miyamoto et al., 1999; Niino et al., 1987; Pan et al., 2003; Shiota et al., 1997).

In this study, it will be solved a one dimensional diffusion equation in FGMs

$$c(x)\rho(x)\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(k(x)\frac{\partial T}{\partial x}\right) \quad (1)$$

with variable properties like conductivity and heat capacity per unit volume. The compositional variation in FGM will be assumed as an exponential function by defining the conductivity and heat capacity per unit volume, respectively, as

$$\begin{aligned} &k(x)=k_0e^{\alpha x}, \quad c(x)\rho(x)=c_0\rho_0e^{-\alpha x} \\ &\alpha>0, \qquad a\le x\le b \end{aligned} \quad (2)$$

and as a power function by defining the conductivity and heat capacity per unit volume as

$$\begin{aligned} &k(x)=k_0(1+x)^{\alpha}, \quad c(x)\rho(x)=c_0\rho_0(1+x)^{-\alpha} \\ &\alpha>0,\ \ \alpha\ne 1, \qquad a\le x\le b \end{aligned} \quad (3)$$

where the parameter α is named as a nonhomogeneity parameter of the FGM and parameters k_0, c_0, ρ_0 are constants. It will be employed Adomian decomposition method to solve the problem in which the corresponding eigenfunctions of the related eigenvalue problem are expressed in terms of generalized Fourier series.

METHODOLOGY

Let us consider the general deterministic functional equation

$$Lu+Ru+Nu=f(t) \quad (4)$$

where L is a linear operator which is invertible under sufficient existence and regularity conditions so that L^{-1} exists. Also, when L^{-1} is applied on a function $f(t)$ then $L^{-1}f(t)$ is measurable. R is the remainder of the linear operator and N represents a nonlinear operator. Applying L^{-1} on both sides of (4), it can be obtained that:

$$L^{-1}(Lu)=L^{-1}(f(t))-L^{-1}(Ru)-L^{-1}(Nu) \quad (5)$$

Where L^{-1} is an integral operator, $\int_{t_0}^{t}(\cdot)d\tau$, and the result can be simplified as:

$$u(t)=u(t_0)+L^{-1}(f(t))-L^{-1}(Ru)-L^{-1}(Nu) \quad (6)$$

The standard ADM defines the solution in the form of

$$u=\sum_{n=0}^{\infty}u_n \quad (7)$$

where the components $u_n, (n=0,1,2,\dots)$, are determined recursively by using the relation

$$u_0=u(t_0)+L^{-1}(f(t)) \quad (8)$$

$$u_{n+1}=-L^{-1}(Ru_n)-L^{-1}(Nu_n), \quad n\ge 0 \quad (9)$$

The nonlinear term Nu_n can be represented by an infinite series of the form

$$Nu=\sum_{n=0}^{\infty}A_n \quad (10)$$

where $A_n, (n=0,1,2,\mathrm{K})$, are the appropriate Adomian's polynomials which are given by

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right]_{\lambda=0}, \quad n\ge 0. \quad (11)$$

If the series converges in a suitable way, then the general solution is obtained as

$$u(x)=\lim_{n\to\infty}\sum_{k=0}^{n}u_k(x) \quad (12)$$

Conductivity as an Exponential Function

Let us consider the diffusion equation

$$e^{-\alpha x}\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(k\,e^{\alpha x}\frac{\partial T}{\partial x}\right), \qquad k=\frac{k_0}{c_0\rho_0} \quad (13)$$

with boundary and initial conditions, respectively,

$$T(a,t)=0, \quad T(b,t)=\eta t, \quad t>0, \\ T(x,0)=0, \quad a\le x\le b \tag{14}$$

where η is constant. The solution of the Equation (13) can be obtained by proposing a solution such as

$$T(x,t)=u(x,t)+v(x,t) \tag{15}$$

in which the function $v(x,t)$ satisfies the homogeneous equation

$$\frac{d}{dx}\left(e^{\alpha x}\frac{dv}{dx}\right)=0 \tag{16}$$

The solution of (16) along with the nonhomogeneous boundary conditions

$$v(a,t)=0, \quad v(b,t)=\eta t, \quad t>0 \tag{17}$$

is given by

$$v(x,t)=\frac{e^{-\alpha x}-e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}}\eta t \tag{18}$$

Using (18) in (4), it can be obtained a new nonhomogeneous PDE

$$\frac{\partial u}{\partial t}=k\,e^{\alpha x}\frac{\partial}{\partial x}\left(e^{\alpha x}\frac{\partial u}{\partial x}\right)-\eta\frac{e^{-\alpha x}-e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}} \tag{19}$$

with homogeneous boundary and initial conditions, respectively,

$$u(a,t)=0, \quad u(b,t)=0, \quad t>0 \\ u(x,0)=0, \quad a\le x\le b. \tag{20}$$

At this point, it will be used the Adomian decomposition method to solve the problem. First, let us write the equation in operator form as follows:

$$L_t u = L_x u - F(x) \tag{21}$$

Where

$$L_t=\frac{\partial}{\partial t}, \quad L_x=ke^{\alpha x}\frac{\partial}{\partial x}\left(e^{\alpha x}\frac{\partial}{\partial x}\right), \\ F(x)=\frac{\partial v}{\partial t}=\eta\frac{e^{-\alpha x}-e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}}. \tag{22}$$

However, if ADM is applied to (19) directly, $u_1(x,t)=0$ is obtained which yields the wrong general solution due to the iterative nature of ADM in (8) and (9). To fix this, the function $F(x)$ will be expressed as a generalized Fourier series in terms of the eigenfunction expansions of the Sturm-Liouville boundary value problem

$$\frac{d}{dx}\left(e^{\alpha x}\frac{d}{dx}\psi(x)\right)+\frac{\mu^2}{e^{\alpha x}}\psi(x)=0 \tag{23}$$

with homogeneous boundary conditions $y(a)=y(b)=0$. Eigenvalues and the corresponding eigenfunctions can be easily obtained as

$$\mu^2=\frac{n^2\pi^2\alpha^2}{\left(e^{-\alpha b}-e^{-\alpha a}\right)^2}, \\ \psi_n(x)=c_n\sin\left(\frac{e^{-\alpha x}-e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}}n\pi\right)$$

Where

$$c_n=\sec\left(\frac{e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}}n\pi\right), \qquad n=1,2,3,\dots .$$

Finally, the function $F(x)$ can be expressed as

$$F(x)=\sum_{n=1}^{\infty}f_n\psi_n(x), \quad =\sum_{n=1}^{\infty}f_n c_n\sin\left(\frac{e^{-\alpha x}-e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}}n\pi\right) \tag{24}$$

Where f_n is called as coefficients of generalized Fourier series of $F(x)$ and using orthogonal properties of eigenfunctions it is evaluated that

$$f_n=\frac{2}{\pi c_n}\int_a^b F(x)\sin\left(\frac{e^{-\alpha x}-e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}}n\pi\right)\frac{-\alpha\pi e^{-\alpha x}}{e^{-\alpha b}-e^{-\alpha a}}dx, \\ f_n=-(-1)^n\frac{2\eta}{n\pi c_n}.$$

Now, defining the solution of $u(x,t)$ and the inverse of operator L_t, respectively, as

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t), \quad L_t^{-1}=\int_0^t(\cdot)dt \tag{25}$$

then, the ADM method can be applied to (19) as follows:

$$L_t^{-1}(L_t u)=L_t^{-1}(L_x u)-L_t^{-1}(F(x)), \tag{26}$$

$$u(x,t)=u(x,0)-L_t^{-1}(F(x))+L_t^{-1}(L_x u). \tag{27}$$

Let us write that

$$u_0 = u(x,0) - L_t^{-1}(F(x)),$$
$$u_0 = -t\sum_{n=1}^{\infty} f_n c_n \sin\left(\frac{e^{-\alpha x} - e^{-\alpha a}}{e^{-\alpha b} - e^{-\alpha a}} n\pi\right). \quad (28)$$

From the recursive relation

$$u_{j+1}(x,t) = L_t^{-1}(L_x u_j), \qquad j = 0,1,2,\ldots,$$

it can be obtained that

$$u_1(x,t) = L_t^{-1}(L_x u_0)$$
$$u_1(x,t) = \int_0^t k e^{\alpha x}\frac{\partial}{\partial x}\left(e^{\alpha x}\frac{\partial}{\partial x}\right.$$
$$\left.\times\left[-t\sum_{n=1}^{\infty} f_n c_n \sin\left(\frac{e^{-\alpha x} - e^{-\alpha a}}{e^{-\alpha b} - e^{-\alpha a}} n\pi\right)\right]\right) \quad (29)$$

$$u_2(x,t) = L_t^{-1}(L_x u_1),$$
$$u_2(x,t) = -\frac{t^3}{6}\sum_{n=1}^{\infty} f_n c_n \left(\frac{k n^2\pi^2\alpha^2}{\left(e^{-\alpha b} - e^{-\alpha a}\right)^2}\right)^2$$
$$\times\sin\left(\frac{e^{-\alpha x} - e^{-\alpha a}}{e^{-\alpha b} - e^{-\alpha a}} n\pi\right) \quad (30)$$

$$u_3(x,t) = L_t^{-1}(L_x u_2),$$
$$u_3(x,t) = -\frac{t^4}{24}\sum_{n=1}^{\infty} f_n c_n \left(\frac{k n^2\pi^2\alpha^2}{\left(e^{-\alpha b} - e^{-\alpha a}\right)^2}\right)^3$$
$$\times\sin\left(\frac{e^{-\alpha x} - e^{-\alpha a}}{e^{-\alpha b} - e^{-\alpha a}} n\pi\right) \quad (31)$$

and so on. Finally, the j^{th} component of the solution can be given as

$$u_j(x,t) = L_t^{-1}(L_x u_{j-1}),$$
$$u_j(x,t) = (-1)^{j+1}\frac{t^{j+1}}{(j+1)!}\sum_{n=1}^{\infty} f_n c_n \left(\frac{k n^2\pi^2\alpha^2}{\left(e^{-\alpha b} - e^{-\alpha a}\right)^2}\right)^j$$
$$\times\sin\left(\frac{e^{-\alpha x} - e^{-\alpha a}}{e^{-\alpha b} - e^{-\alpha a}} n\pi\right). \quad (32)$$

From the superposition of $u_j(x,t)$ and using the identity

$$(-1)^{j+1}\frac{t^{j+1}}{(j+1)!} = -t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \ldots,$$

$$(-1)^{j+1}\frac{t^{j+1}}{(j+1)!} = e^{-t} - 1 \quad (33)$$

the solution of $u(x,t)$ can be written as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$
$$= \sum_{n=1}^{\infty} f_n c_n \sin\left(\frac{e^{-\alpha x} - e^{-\alpha a}}{e^{-\alpha b} - e^{-\alpha a}} n\pi\right)\left(\frac{e^{-k\theta t} - 1}{k\theta}\right) \quad (34)$$

Where

$$\theta = \frac{n^2\pi^2\alpha^2}{\left(e^{-\alpha b} - e^{-\alpha a}\right)^2} \quad \text{and} \quad f_n = -(-1)^n\frac{2\eta}{n\pi c_n}. \quad (35)$$

The general solution of the diffusion equation with exponentially varying conductivity from (15) is given by

$$T(x,t) = \frac{e^{-\alpha x} - e^{-\alpha a}}{e^{-\alpha b} - e^{-\alpha a}}\eta t$$
$$+\sum_{n=1}^{\infty} f_n \sin\left(\frac{e^{-\alpha x} - e^{-\alpha a}}{e^{-\alpha b} - e^{-\alpha a}} n\pi\right)\left(\frac{e^{-k\theta t} - 1}{k\theta}\right). \quad (36)$$

Conductivity as a Power Function

Let us consider the diffusion equation in the form of

$$(1+x)^{-\alpha}\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k(1+x)^{\alpha}\frac{\partial T}{\partial x}\right), \quad k = \frac{k_0}{c_0\rho_0} \quad (37)$$

with the same boundary and initial conditions given in (14). Proposing the same solution in (15), the function $v(x,t)$ satisfies the equation

$$\frac{d}{dx}\left((1+x)^{\alpha}\frac{dv}{dx}\right) = 0. \quad (38)$$

The solution of (38) along with the non-homogeneous boundary conditions given in (17) can be obtained as:

$$v(x,t)=\frac{(1+x)^{1-\alpha}-(1+a)^{1-\alpha}}{(1+b)^{1-\alpha}-(1+a)^{1-\alpha}}\eta t. \tag{39}$$

Finally, substituting $v(x,t)$ into (15), non-homogeneous diffusion equation is obtained

$$\frac{\partial u}{\partial t}=k(1+x)^{\alpha}\frac{\partial}{\partial x}\left((1+x)^{\alpha}\frac{\partial u}{\partial x}\right)-\frac{\partial v}{\partial t} \tag{40}$$

with homogeneous boundary and initial conditions given in (20), respectively, and then it can be written in an operator form given in (21) in which operators and the function $F(x)$ can be defined as follows:

$$L_t=\frac{\partial}{\partial t},\qquad L_x=k(1+x)^{\alpha}\frac{\partial}{\partial x}\left((1+x)^{\alpha}\frac{\partial u}{\partial x}\right),$$
$$F(x)=\frac{\partial v}{\partial t}=\eta\frac{(1+x)^{1-\alpha}-(1+a)^{1-\alpha}}{(1+b)^{1-\alpha}-(1+a)^{1-\alpha}}. \tag{41}$$

Again, solving the boundary value problem like

$$\frac{d}{dx}\left((1+x)^{\alpha}\frac{d}{dx}\psi(x)\right)+\frac{\mu^2}{(1+x)^{\alpha}}\psi(x)=0 \tag{42}$$

with boundary conditions $\psi(a)=\psi(b)=0$, the function $F(x)$ can be expressed as a generalized Fourier series

$$F(x)=\sum_{n=1}^{\infty}f_n\psi_n(x) \tag{43}$$

Where

$$f_n=\frac{2}{\pi c_n}\int_a^b F(x)\sin\left(\frac{(1+x)^{1-\alpha}-(1+a)^{1-\alpha}}{(1+b)^{1-\alpha}-(1+a)^{1-\alpha}}n\pi\right)$$
$$\times\frac{(1-\alpha)\pi(1+x)^{-\alpha}}{(1+b)^{1-\alpha}-(1+a)^{1-\alpha}}dx$$

and eigenvalues and the corresponding eigenfunctions are given as

$$\mu^2=\frac{(\alpha-1)^2n^2\pi^2}{\left((1+b)^{1-\alpha}-(1+a)^{1-\alpha}\right)^2},$$

$$\psi_n(x)=c_n\sin\left(\frac{(1+x)^{1-\alpha}-(1+a)^{1-\alpha}}{(1+b)^{1-\alpha}-(1+a)^{1-\alpha}}n\pi\right),$$

$$c_n=\sec\left(\frac{n\pi(1+a)^{1-\alpha}}{(1+b)^{1-\alpha}-(1+a)^{1-\alpha}}\right),\qquad n=1,2,3,\ldots.$$

Defining the unknown function $u(x,t)$ and the inverse operator as in (25), it can be applied the ADM to (40) and it will be obtained the first term in the solution series such that

$$u_0=-t\sum_{n=1}^{\infty}f_nc_n\sin\left(\frac{(1+x)^{1-\alpha}-(1+a)^{1-\alpha}}{(1+b)^{1-\alpha}-(1+a)^{1-\alpha}}n\pi\right). \tag{44}$$

From the Adomian recursive relation the j^{th} component of the solution can be given as

$$u_j(x,t)=L_t^{-1}(L_xu_{j-1}),$$
$$=(-1)^{j+1}\frac{t^{j+1}}{(j+1)!}\sum_{n=1}^{\infty}f_nc_n\left(\frac{k(1-\alpha)^2n^2\pi^2}{\left((1+b)^{1-\alpha}-(1+a)^{1-\alpha}\right)^2}\right)^j$$
$$\times\sin\left(\frac{(1+x)^{1-\alpha}-(1+a)^{1-\alpha}}{(1+b)^{1-\alpha}-(1+a)^{1-\alpha}}n\pi\right). \tag{45}$$

The general solution of the diffusion equation with power variation using the identity given in (33) can be expressed in the form of

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)$$
$$u(x,t)=\sum_{n=1}^{\infty}f_nc_n\sin\left(\frac{(1+x)^{1-\alpha}-(1+a)^{1-\alpha}}{(1+b)^{1-\alpha}-(1+a)^{1-\alpha}}n\pi\right)$$
$$\times\left(\frac{e^{-k\theta t}-1}{k\theta}\right) \tag{46}$$

Where

$$\theta=\frac{(1-\alpha)^2n^2\pi^2}{\left((1+b)^{1-\alpha}-(1+a)^{1-\alpha}\right)^2}, \tag{47}$$

The general solution of the diffusion equation with power form of the conductivity given by

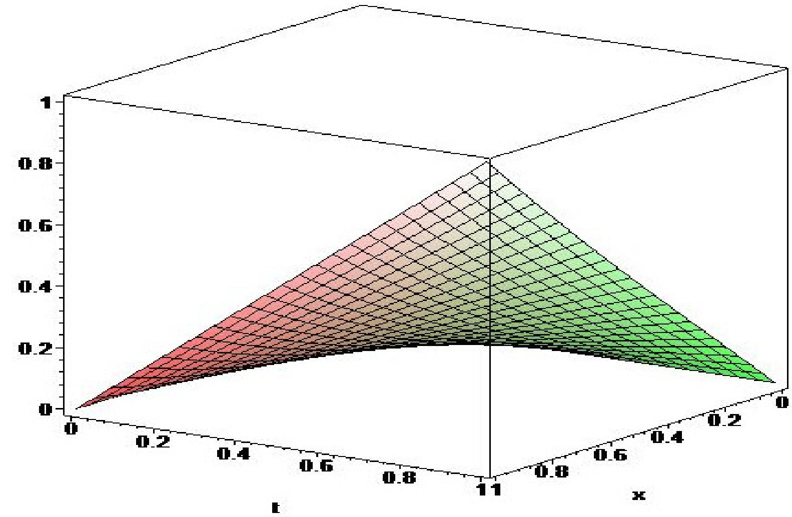

Figure 1. T (×, t) for exponential form of conductivity when α = 0.25.

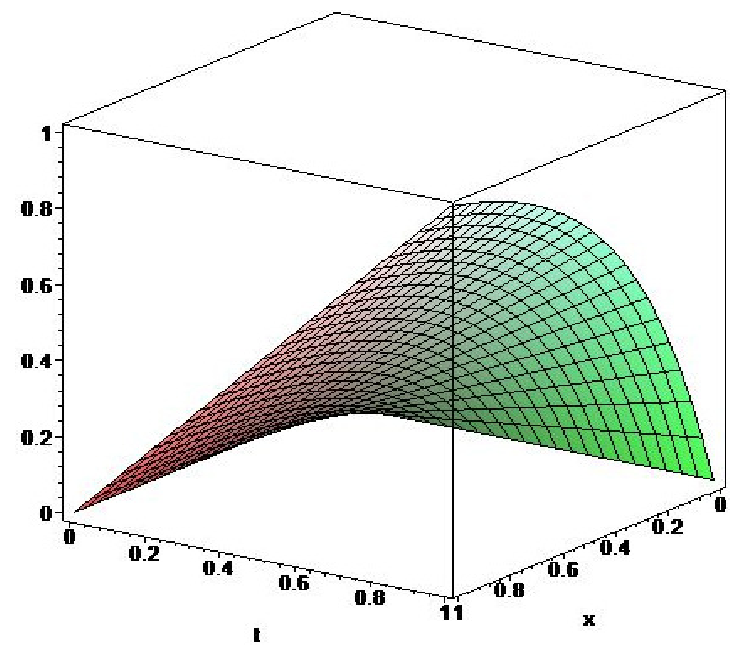

Figure 2. $T(x,t)$ for exponential form of conductivity when α = 3.0.

$$T(x,t)=\frac{(1+x)^{1-\alpha}-(1+a)^{1-\alpha}}{(1+b)^{1-\alpha}-(1+a)^{1-\alpha}}\eta t+\sum_{n=1}^{\infty} f_n c_n \sin\left(\frac{(1+x)^{1-\alpha}-(1+a)^{1-\alpha}}{(1+b)^{1-\alpha}-(1+a)^{1-\alpha}}n\pi\right)\times\left(\frac{e^{-k\theta t}-1}{k\theta}\right). \quad (48)$$

RESULTS

Figures 1 and 2 show the effect of α for the exponentially varying conductivity. As it is seen that the heat conduction is rapidly increasing with increasing non-homogeneity parameter α. In Figures 3 and 4, the effects of $\alpha, (\alpha \neq 1)$ for the conductivity in the form of power function are shown. Either exponential function or power function it can be seen that the effect of the definition of conductivity on heat conduction is slightly different. This is because of the solution $v(x,t)$ that is the dominant

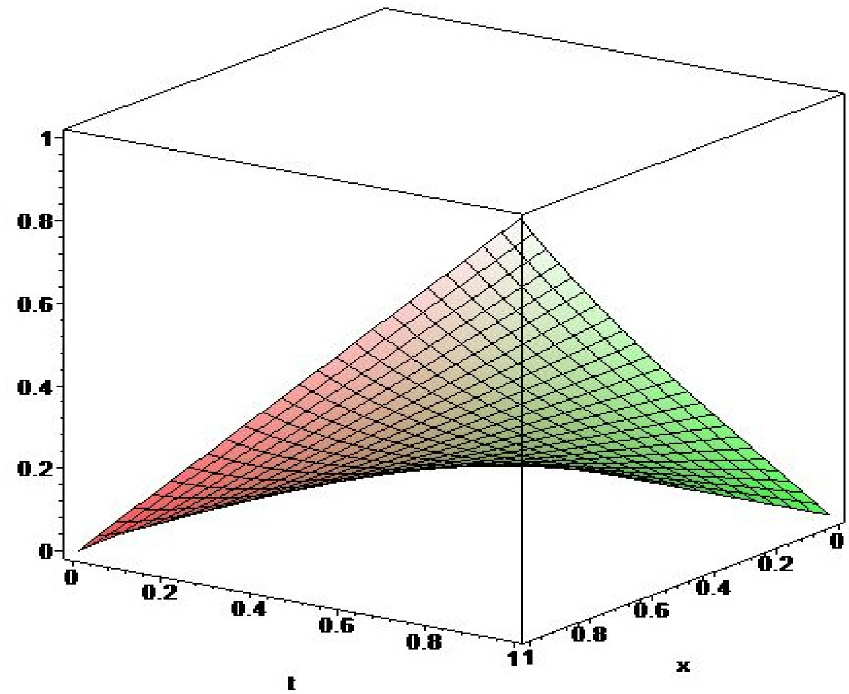

Figure 3. T (×, t) for power form of conductivity when α = 0.25.

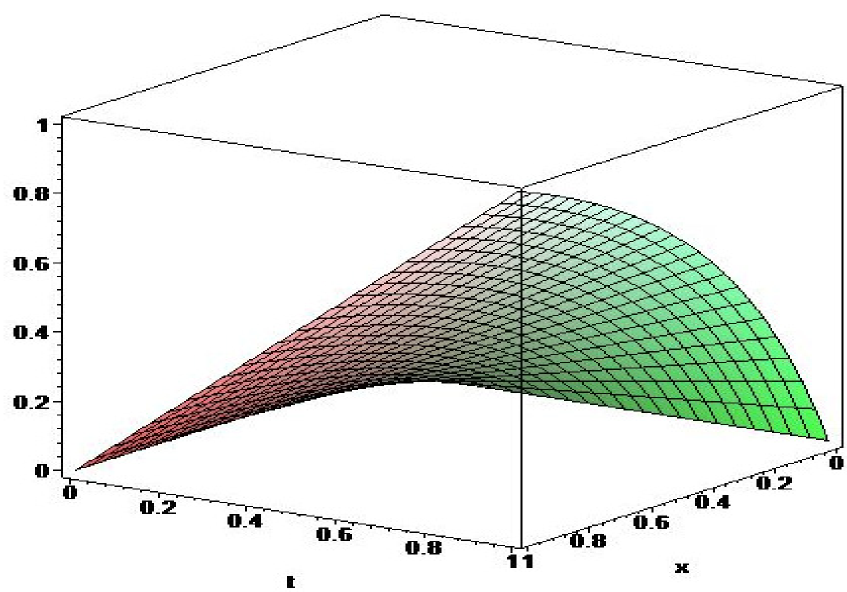

Figure 4. T (×, t) for power form of conductivity when α = 3.0.

term of the series solution of the problem as $n \to \infty$.

Conclusion

In this study, the heat conduction in a non-homogeneous composite material where the conductivity isgiven as an exponential function or a power function is solved using ADM. For the exponentially varying conductivity, the results that are obtained by ADM and eigenfunction expansion method are in complete agreement as shown in Appendix. In addition, it can be shown that the exact solution of the diffusion equation with conductivity as a power function can be obtained by the same method. In Figures, the effect of the non-homogeneity parameter α is shown for the fixed values of

$\eta = 1.0,\ k_0 = 1.0,\ a = 0,\ b = 1.0$.

REFERENCES

Adomian G (1988). A review of the decomposition method in applied mathematics, J. Math. Anal. Appl., 135: 501–544.

Adomian G (1989). Nonlinear stochastic systems and applications to physics, Dordrecht: Kluwer Academic Publishers Group.

Adomian G (1994). Solving frontier problems of physics: The decomposition method, Kluwer Academic Publishers, Boston.

Aslanov A (2009). Approximate solutions of Emden–Fowler type equations, Int. J. Comput. Math., 86(5): 807-826.

Cherruault Y (1989). Convergence of Adomian's method, Kybernetes. 18: 31-38.

Gorguis A, Benny Chan WK (2008). Heat equation and its comparative solutions, Comput. Math. Appl., 55: 2973-2980.

Jang B (2007). Exact solutions to one-dimensional non-homogeneous parabolic problems by the homogeneous ADM, Appl. Math. Comput., 186: 969-979.

Lesnic D (2002). Convergence of ADM: Periodic temperature, Comput. Math. Appl., 44: 13-24.

Kaya D, El-Sayed SM (2003a). An application of the decomposition method for the generalized KdV and RLW equations, Chaos Solitons Fractals, 17: 869-877.

Kaya D, El-Sayed SM (2003b). On a generalized fifth order KdV equations, Phys. Lett. A., 310: 44-51.

Kaysser WA (1999). Material science forum–functionally graded materials, Trans Tech Publications, Enfield, New Hampshire. pp. 308-311.

Marwat KDN, Asghar S (2008). Solution of the heat equation with variable properties by two-step ADM, Math. Comput. Model., 48: 83-90.

Momani S (2005). A numerical scheme for the solution of Sivashinsky equation, Appl. Math. Comput., 168: 1273-1280.

Miyamoto M, Kaysser WA, Rabin BH (1999). A. Kawasaki, R.G. Ford (Eds.), Functionally graded materials: Design, processing and applications. Kluwer Academic Publishers. Norwell, Massachusetts. Niino M, Hirai T, Watanabe R (1987). The functionally gradient materials. J. Jpn. Soc. Comput. Mater., 13: 257-264.

Pan W, Gong J, Zhong L, Chen L (eds) (2003). Functionally graded materials VII. Trans Tech Publications LTD. Switzerland.

Shiota I, Miyamoto Y (1997). FGM-96, Proc. 4th Int. symp. functionally gradient materials. Elsevier Science. Amsterdam, the Netherlands.

Wazwaz AM (1999a). The modified decomposition method and the Pade approximants for solving Thomas–Fermi equation, Appl. Math. Comput. 105: 11-19.

Wazwaz AM (1999b). A reliable modification of Adomian decomposition method, Appl. Math. Comput. 102: 77-86.

Wazwaz AM (2001). A new method for solving differential equations of the Lane–Emden type, Appl. Math. Comput., 118: 287-310.

Appendix

Eigenfunction expansion solution for exponentially varying conductivity

Let us solve the problem in (13) along with the boundary conditions in (14) by the method of eigenfunction expansion. Using the same steps from (13) through (18), it is obtained the nonhomogeneous PDE.

$$\frac{\partial u}{\partial t}=ke^{\alpha x}\frac{\partial}{\partial x}\left(e^{\alpha x}\frac{\partial u}{\partial x}\right)-\eta\frac{e^{-\alpha x}-e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}} \tag{A.1}$$

with homogeneous boundary and initial conditions, respectively,

$$\begin{aligned} &u(a,t)=0, \quad u(b,t)=0, \quad t>0 \\ &u(x,0)=0, \quad a\le x\le b. \end{aligned} \tag{A.2}$$

The eigenfunctions of the related homogeneous problem are given by

$$\frac{d}{dx}\left(e^{\alpha x}\frac{d}{dx}\psi(x)\right)+\frac{\mu^2}{e^{\alpha x}}\psi(x)=0 \tag{A.3}$$

with homogeneous boundary conditions $y(a)=y(b)=0.$ The eigenvalues and the corresponding eigenfunctions are

$$\mu^2=\frac{n^2\pi^2\alpha^2}{\left(e^{-\alpha b}-e^{-\alpha a}\right)^2}, \qquad \psi_n(x)=c_n\sin\left(\frac{e^{-\alpha x}-e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}}n\pi\right) \tag{A.4}$$

where

$$c_n=\sec\left(\frac{e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}}n\pi\right), \qquad n=1,2,3,\ldots.$$

By defining the solution as

$$u(x,t)=\sum_{n=1}^{\infty}A_n(t)\psi_n(x)=\sum_{n=1}^{\infty}A_n(t)c_n\sin\left(\frac{e^{-\alpha x}-e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}}n\pi\right) \tag{A.5}$$

and substituting in (A.1) it is obtained

$$\sum_{n=1}^{\infty}\frac{dA_n(t)}{dx}\psi_n(x)=ke^{\alpha x}\frac{d}{dx}\left(e^{\alpha x}\sum_{n=1}^{\infty}A_n(t)\frac{d\psi_n(x)}{dx}\right)-\eta\frac{e^{-\alpha x}-e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}},$$

$$\sum_{n=1}^{\infty}\frac{dA_n(t)}{dx}\psi_n(x)-\sum_{n=1}^{\infty}A_n(t)\left(\frac{d^2\psi_n(x)}{dx^2}+\alpha\frac{d\psi_n(x)}{dx}\right)ke^{2\alpha x}=-\eta\frac{e^{-\alpha x}-e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}},$$

$$\sum_{n=1}^{\infty}\left(\frac{dA_n(t)}{dx}+\frac{kn^2\pi^2\alpha^2}{\left(e^{-\alpha b}-e^{-\alpha a}\right)^2}A_n(t)\right)c_n\sin\left(\frac{e^{-\alpha x}-e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}}n\pi\right)=-\eta\frac{e^{-\alpha x}-e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}}.$$

Let Let $\frac{e^{-\alpha x}-e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}}\pi=\xi$ and using orthogonality

$$\sum_{n=1}^{\infty}\left(\frac{dA_n(t)}{dx}+\frac{kn^2\pi^2\alpha^2}{\left(e^{-\alpha b}-e^{-\alpha a}\right)^2}A_n(t)\right)c_n\int_0^{\pi}\sin(n\xi)\sin(m\xi)\,d\xi=-\int_0^{\pi}\eta\frac{\xi}{\pi}\sin(m\xi)\,d\xi,$$

it is obtained an ordinary first order differential equation to be solved

$$\frac{dA_n(t)}{dx}+\frac{kn^2\pi^2\alpha^2}{\left(e^{-\alpha b}-e^{-\alpha a}\right)^2}A_n(t)=(-1)^n\frac{2\eta}{n\pi c_n}. \tag{A.6}$$

The solution can be obtained straightforward as

$$A_n(t)=-(-1)^n\frac{2\eta}{n\pi c_n}\frac{\left(e^{-\alpha b}-e^{-\alpha a}\right)^2}{kn^2\pi^2\alpha^2}\left(e^{\frac{-kn^2\pi^2\alpha^2}{\left(e^{-\alpha b}-e^{-\alpha a}\right)^2}}-1\right). \tag{A.7}$$

and using definitions in (35), it can be written that

$$A_n(t)=f_n\left(\frac{e^{-k\theta}-1}{k\theta}\right) \tag{A.8}$$

where f_n is defined in (25). The general solution in terms of eigenfunction expansion method of the problem given in (13) is found as

$$T(x,t)=v(x,t)+u(x,t)=\frac{e^{-\alpha x}-e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}}\eta t+\sum_{n=1}^{\infty}f_n\sin\left(\frac{e^{-\alpha x}-e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}}n\pi\right)\left(\frac{e^{-k\theta t}-1}{k\theta}\right). \tag{A.9}$$