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Simplified synchronization criteria for complex dynamical networks with time-varying delays

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This paper proposes new delay-dependent synchronization criteria for complex dynamical networks with time-varying delays. By constructing a suitable Lyapunov-Krasovskii's functional and utilizing Finsler's lemma, a novel synchronization criterion for the networks were established in terms of linear matrix inequalities (LMIs) which can be easily solved by various effective optimization algorithms. Numerical examples were given to illustrate the effectiveness of the proposed methods.

Key words: Complex networks, time-varying delays, synchronization, Lyapunov method, linear matrix inequalities (LMIs).

INTRODUCTION

During the last few years, complex dynamic networks (CDNs), which are a set of interconnected nodes with specific dynamics, have received increasing attention from the real world such as the internet, the world wide web (WWW), social networks, electrical power grids, global economic markets, and so on. Many mathematical models were proposed to describe various complex networks, small-world networks and scale-free networks (Watts et al., 1998; Strogatz, 2001; Boccaletti et al., 2006). In the implementation of many practical CDNs, there exists time-delay because of the finite information processing speed. It is well known that time-delay often causes undesirable dynamic behaviors such as oscillation, performance degradation, and instability of the network. Therefore, various approaches to synchronization analysis for CDNs with time-delay has been investigated in the literature (Li et al., 2004, 2008;

Gao et al., 2006; Koo et al., 2010). By using network modeling with coupling delays, Li et al. (2004) proposed the synchronization criteria for the CDNs with time-delay, expressed in the form of LMIs for the first time. In Gao et al. (2006), new delay-dependent synchronization criteria were derived for continuous- and discrete-time delayed networks. In Li et al. (2008), based on free-weighting matrix method, the problem of synchronization for CDNs with time-varying delay was considered. Koo et al. (2010) presented a synchronization criterion for singular CDNs with time-varying delays.

The stability criterion for time-delay system can be classified into two; delay-dependent and delay-independent. Since delays-dependent stability criteria include the information on the size of delay, delays-dependent stability criteria are generally less conservative than delay-independent. Therefore, a great number of results on delay-dependent stability conditions for time-delay systems have been reported in the literature (Niculescu, 2002; Richard, 2003; Gu, 2000; Suplin et al., 2006; Xu and lam, 2007; Kwon et al., 2010).

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On the other hand, among various methods of synchronization for CDNs with time-varying delays, a remarkable one is the free-weighting matrix method used in Li et al. (2008), which is very effective jn tackling the delay-dependent stability analysis for time-delay systems since both bounding techniques and model transformations are not involved. However, the free-weighting matrix method often needs to introduce many decision variables in the stability conditions and leads to a significant increase in the computational burden and time-consuming. Here, one natural question is how to obtain an upper bound of time-delay for guaranteeing synchronization of CDNs with time-varying delays as possible as large by employing fewer decision variables. With this motivation, we propose improved delay-dependent synchronization criteria for CDNs with time-varying delays. Both time varying delays in network coupling and dynamical nodes have been considered. By constructing a suitable Lyapunov-Krasovskii's (L-K) functional and utilizing Finsler's lemma without free-weighting matrices, new stability criterion were derived in terms of LMIs, which can be solved efficiently by using the interior-point algorithms (Boyd et al., 1994). Two numerical examples are included to show the effectiveness of the proposed method.

Notation: $\mathbf{R}^n$ is the n-dimensional Euclidean space, $\mathbf{R}^{m \times n}$ denotes the set of $m \times n$ real matrix. For symmetric matrices X and Y; $X > Y$ (respectively, $X \geq Y$) means that the matrix $X - Y$ is positive definite (respectively, nonnegative). $I_n \in \mathbf{R}^{n \times n}$ and $0_n \in \mathbf{R}^{n \times n}$ denote the n-dimensional identity matrix and zero matrix, respectively. $0_{m \times n} \in \mathbf{R}^{m \times n}$ denotes the $m \times n$ zero matrix. $\| \cdot \|$ refers to the Euclidean vector norm and the induced matrix norm. $diag\{\cdots\}$ denotes the block diagonal matrix. $*$ represents the elements below the main diagonal of a symmetric matrix. For a given matrix $X \in \mathbf{R}^{m \times n}$, such that $rank(X) = r$, we define $X^{\perp} \in \mathbf{R}^{n \times (n-r)}$ as the right orthogonal complement of X; that is, $XX^{\perp} = 0$. $X_{[h(t)]} \in \mathbf{R}^{m \times n}$ means that the elements of the matrix $X_{[h(t)]}$ includes the value of $h(t)$; for example, $X_{[h_U]} \equiv X_{[h(t)=h_U]}$.

Problem statements

Consider the following CDNs with both time-varying delays in network coupling and in nodes for $i = 1, ..., N$

$$\dot{y}_i(t) = f(y_i(t), y_i(t - h(t))) + c \sum_{j=1}^{N} g_{ij} A y_j(t - h(t)), \quad (1)$$

Where N is the number of couple nodes, $y_i(t) = [y_{i1}(t), ..., y_{in}(t)]^T \in \mathbf{R}^n$ is the state vector of the i th node, $f(\cdot) \in \mathbf{R}^n$ is a continuous differentiable vector function, the constant c is the coupling strength, and $h(t)$ is a time-varying coupling delay satisfying;

$$0 \leq h(t) \leq h_M, \ \dot{h}(t) \leq h_D, \quad (2)$$

Where h_M is a positive scalar and h_D is any constant one. $A = [a_{ij}]_{n \times n} \in \mathbf{R}^{n \times n}$ is the constant inner-coupling matrix of nodes, of some pairs (i, j), $1 \leq i, j \leq n$, with $a_{ij} \neq 0$, which means two coupled nodes are linked through their i th and j th state variables, otherwise $a_{ij} = 0$, and $G = [g_{ij}]_{n \times n} \in \mathbf{R}^{n \times n}$ is the outer-coupling matrix of the network, in which g_{ij} is defined as follows: if there is a connection between i th and j th node $(j \neq i)$, then $g_{ij} = g_{ji} = 1$; otherwise, $g_{ij} = g_{ji} = 0$ $(j \neq i)$, and the diagonal elements of matrix G are defined by;

$$g_{ii} = -\sum_{j=1, i \neq j}^{N} g_{ij} = -\sum_{j=1, i \neq j}^{N} g_{ji}, \ i = 1, ..., N. \quad (3)$$

It is assumed that the network (1) is connected in a way that there is no isolated cluster, that is, G is an irreducible matrix.

The goal of this paper is to investigate the delay-dependent synchronization analysis (in other words, stability analysis) of CDNs with time-varying delays (1). In order to do this, we introduce the following definition and lemmas.

Definition 1

According to Li et al. (2004) the delayed dynamical networks (1) is said to achieve asymptotic synchronization if;

$$y_1(t) = y_2(t) = ... = y_N(t) = s(t) \ as \ t \to \infty, \quad (4)$$

Where $s(t) \in \mathbf{R}^n$ is a solution of an isolated node, satisfying $\dot{s}(t) = f(s(t), s(t - h(t)))$.

Lemma 1

(Li et al., 2004) considered the network (1). Let $0 = \lambda_1 > \lambda_2 \geq ... \geq \lambda_N$ be the eigenvalues of the outer-coupling matrix G. If the following $N - 1$ linear time varying delayed, differential equations are;

$$\dot{x}_k(t) = J(t)x_k(t) + (J_d(t) + c\lambda_k A)x_k(t - h(t)), \quad (5)$$

Where $J(t) \in \mathbf{R}^{n \times n}$ and $J_d(t) \in \mathbf{R}^{n \times n}$ are the Jacobian of $f(s(t), s(t - h(t)))$ at $s(t)$ and $s(t - h(t))$, respectively. Then the synchronized states (4) are asymptotically stable.

Proof: Follow the proof of Theorem 1 in Li et al. (2004).

From definition 1 and lemma 1, the problem of synchronization for system (1) can be considered as asymptotic stability analysis of system (5). The following lemmas will be used to obtain our proposed synchronization criteria.

Lemma 2: (Finsler's Lemma (de Oliveira et al., 2001)) Let $\zeta \in \mathbf{R}^n$, $\Phi = \Phi^T \in \mathbf{R}^{n \times n}$, and $B \in \mathbf{R}^{m \times n}$ such that $rank(B) < n$. The following statements are equivalent:

(i) $\zeta^T \Phi \zeta < 0, \forall B \zeta = 0, \zeta \neq 0,$

(ii) $(B^\perp)^T \Phi (B^\perp) < 0,$

(iii) $\exists X \in \mathbf{R}^{n \times m} : \Phi + XB + B^T X^T < 0.$

Lemma 3: (Köse et al., 1998).

Let $M(h)$ be affinely dependent on $h \in H = \{h(t) : \underline{h}_i \leq h_i(t) \leq \bar{h}_i, \forall i = 1,2,...,s\}$; Then there exist parameters a_k, where $a_k > 0$ for all $k = 1,2,...2^s$ and $\sum_{k=1}^{2^s} a_k = 1$ such that $M(h)$ can be expressed as a convex combination of the vertex value as follows:

$$M(h) = \sum_{k=1}^{2^s} a_k M(h^k). \tag{6}$$

Where h^k are vertex values of $h(t)$ in the set H.

Lemma 4: For any constant matrix $M = M^T > 0$, the following inequality holds:

$$-h(t) \int_{t-h(t)}^{t} \dot{x}^T(s) M \dot{x}(s) ds$$
$$\leq \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} -M & M \\ * & -M \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}. \tag{7}$$

Proof: According to Jensen's inequality in Gu et al. (2000), one obtains:

$$(b-a)\int_b^a \dot{x}^T(s) M \dot{x}(s) ds \leq -\left(\int_b^a \dot{x}(s) ds\right)^T M \left(\int_b^a \dot{x}(s) ds\right). \tag{8}$$

If we choose $a = t$ and $b = t - h(t)$ in (8), inequality (7) can be obtained.

RESULTS

In this section, new synchronization criteria for network (1) will be proposed. Before introducing our main results, the notations of several matrices are defined for simplicity:

$$\zeta_k^T(t) = [x_k^T(t), x_k^T(t-h(t)), x_k^T(t-h_M), \dot{x}_k^T(t)],$$
$$B_k = [J(t), (J_d(t) + c\lambda_k A), 0_n, -I_n],$$
$$\Sigma_{11} = Q_{1k} + Q_{2k} - 2R_k, \quad \Sigma_{22} = -(1-h_D)Q_{2k} - 3R_k$$

$$\Phi_{1k} = \begin{bmatrix} \Sigma_{11} & 2R_k & 0_n & P_k \\ * & \Sigma_{22} & R_k & 0_n \\ * & * & -Q_{1k} - R_k & 0_n \\ * & * & * & h_M^2 R_k \end{bmatrix},$$

$$\Phi_{2k} = \begin{bmatrix} \dfrac{1}{h_M} R_k & -\dfrac{1}{h_M} R_k & 0_n & 0_n \\ * & 0_n & \dfrac{1}{h_M} R_k & 0_n \\ * & * & -\dfrac{1}{h_M} R_k & 0_n \\ * & * & * & 0_n \end{bmatrix},$$

$$\Phi_{k[h(t)]} = \Phi_{1k} + h(t)\Phi_{2k}. \tag{9}$$

Now, a delay-dependent synchronization criteria for system (1) is given as follows:

Theorem: For given scalars $0 < h_M$ and h_D, the network (1) is asymptotically synchronized for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq h_D$, if there exist positive definite matrices $P_k \in \mathbf{R}^{n \times n}$, $Q_{1k} \in \mathbf{R}^{n \times n}$, $Q_{2k} \in \mathbf{R}^{n \times n}$ and $R_k \in \mathbf{R}^{n \times n}$ satisfying the following LMIs for $k = 2,...,N$:

$$(B_k^\perp)^T \Phi_{k[0]} (B_k^\perp) < 0,$$
$$(B_k^\perp)^T \Phi_{k[h_M]} (B_k^\perp) < 0, \tag{10}$$

Where B_k and $\Phi_{k[h(t)]}$ are defined (9).

Proof: Let us consider the following L-K functional candidate as;

$$V_k = V_{1k} + V_{2k} + V_{3k}, \tag{11}$$

Where;

$$V_{1k} = x_k^T(t) P_k x_k(t),$$
$$V_{2k} = \int_{t-h_M}^{t} x_k^T(s) Q_{1k} x_k(s) ds + \int_{t-h(t)}^{t} x_k^T(s) Q_{2k} x_k(s) ds,$$
$$V_{3k} = h_M \int_{t-h_M}^{t} \int_s^t \dot{x}_k^T(u) R_k \dot{x}_k(u) du ds$$

The time-derivative of V_k can be calculated as;

$$\dot{V}_{1k} = 2x_k^T(t)P_k\dot{x}_k(t),$$
$$\dot{V}_{2k} \le x_k^T(t)(Q_{1k} + Q_{2k})x_k(t) + x_k^T(t-h_M)Q_{1k}x_k(t-h_M) \qquad (12)$$
$$-(1-h_D)x_k(t-h(t))Q_{2k}x_k(t-h(t)),$$
$$\dot{V}_{3k} = h_M^2\dot{x}_k^T(t)R_k\dot{x}_k(t) - h_M\int_{t-h(t)}^{t}\dot{x}_k^T(s)R_k\dot{x}_k(s)ds$$
$$-h_M\int_{t-h_M}^{t-h(t)}\dot{x}_k^T(s)R_k\dot{x}_k(s)ds.$$

By using $-h_M = -h(t) - (h_M - h(t))$ and Lemma 4, an upper bound of the first integral term of $\dot{V}_{3k}$ can be obtained as;

$$-h_M\int_{t-h(t)}^{t}\dot{x}_k^T(s)R_k\dot{x}_k(s)ds$$
$$= -h(t)\int_{t-h(t)}^{t}\dot{x}_k^T(s)R_k\dot{x}_k(s)ds$$
$$-(h_M - h(t))\int_{t-h(t)}^{t}\dot{x}_k^T(s)R_k\dot{x}_k(s)ds$$
$$\le -h(t)\int_{t-h(t)}^{t}\dot{x}_k^T(s)R_k\dot{x}_k(s)ds$$
$$-\frac{(h_M - h(t))h(t)}{h_M}\int_{t-h(t)}^{t}\dot{x}_k^T(s)R_k\dot{x}_k(s)ds$$
$$\le -\left(\int_{t-h(t)}^{t}\dot{x}_k(s)ds\right)^T R_k\left(\int_{t-h(t)}^{t}\dot{x}_k(s)ds\right)$$
$$-\frac{(h_M - h(t))}{h_M}\left(\int_{t-h(t)}^{t}\dot{x}_k(s)ds\right)^T R_k\left(\int_{t-h(t)}^{t}\dot{x}_k(s)ds\right) \qquad (13)$$
$$= \begin{bmatrix} x_k(t) \\ x_k(t-h(t)) \end{bmatrix}^T \begin{bmatrix} -\delta_1 R_k & \delta_1 R_k \\ * & -\delta_1 R_k \end{bmatrix} \begin{bmatrix} x_k(t) \\ x_k(t-h(t)) \end{bmatrix},$$

Where; $\delta_1 = 2 - (h(t)/h_M)$. With the similar method introduced above, an upper bound of the second integral term of $\dot{V}_{3k}$ can be estimated as;

$$-h_M\int_{t-h_M}^{t-h(t)}\dot{x}_k^T(s)R_k\dot{x}_k(s)ds$$
$$= -h(t)\int_{t-h_M}^{t-h(t)}\dot{x}_k^T(s)R_k\dot{x}_k(s)ds$$
$$-(h_M - h(t))\int_{t-h_M}^{t-h(t)}\dot{x}_k^T(s)R_k\dot{x}_k(s)ds$$
$$\le -\frac{(h_M - h(t))h(t)}{h_M}\int_{t-h_M}^{t-h(t)}\dot{x}_k^T(s)R_k\dot{x}_k(s)ds$$
$$-(h_M - h(t))\int_{t-h_M}^{t-h(t)}\dot{x}_k^T(s)R_k\dot{x}_k(s)ds$$

$$\le \begin{bmatrix} x_k(t-h(t)) \\ x_k(t-h_M) \end{bmatrix}^T \begin{bmatrix} -\delta_2 R_k & \delta_2 R_k \\ * & -\delta_2 R_k \end{bmatrix} \begin{bmatrix} x_k(t-h(t)) \\ x_k(t-h_M) \end{bmatrix}, \qquad (14)$$

Where; $\delta_2 = 1 + (h(t)/h_M)$. Then, an upper bound of $\dot{V}_{3k}$ can be rewritten as;

$$\dot{V}_{3k} \le h_M^2\dot{x}_k^T(t)R_k\dot{x}_k(t) + \begin{bmatrix} x_k(t) \\ x_k(t-h(t)) \\ x_k(t-h_M) \end{bmatrix}^T \Xi \begin{bmatrix} x_k(t) \\ x_k(t-h(t)) \\ x_k(t-h_M) \end{bmatrix}, \qquad (15)$$

Where;

$$\Xi = \begin{bmatrix} -\left(2-\dfrac{h(t)}{h_M}\right)R_k & \left(2-\dfrac{h(t)}{h_M}\right)R_k & 0_n \\ * & -3R_k & \left(1+\dfrac{h(t)}{h_M}\right)R_k \\ * & * & -\left(1+\dfrac{h(t)}{h_M}\right)R_k \end{bmatrix}.$$

From (11)-(15), the time-derivative of V_k has a new upper bound as;

$$\dot{V}_k \le \zeta_k^T(t)\Phi_{k[h(t)]}\zeta_k(t), \qquad (16)$$

Where; $\Phi_{k[h(t)]}$ and $\zeta_k(t)$ are defined in (9). Also, the network (1) with the augmented vector $\zeta_k(t)$ can be rewritten as;

$$B_k\zeta_k(t) = 0, \qquad (17)$$

Where B_k is defined in (9). Therefore, a synchronization condition for network (1) is;

$$\zeta_k^T(t)\Phi_{k[h(t)]}\zeta_k(t) < 0, \quad \text{subject to } B_k\zeta_k(t) = 0, \qquad (18)$$

Furthermore, from Lemma 2 (iii), Equation (18) is equivalent to the following condition;

$$\Phi_{k[h(t)]} + XB_k + B_k^T X^T < 0, \qquad (19)$$

Where X is any matrix. Let us define;

$$a_1(t) = 1 - \frac{h(t)}{h_M}, \ a_2(t) = \frac{h(t)}{h_M}, \qquad (20)$$

$$H = \{H(t) : 0 \le h(t) \le h_M\}. \qquad (21)$$

Then, the vertex set H_{vex} is defined as;

$$H_{vex} = \{\mathbf{H}^i : \mathbf{H}^1 = 0, \ \mathbf{H}^2 = h_M \}. \tag{22}$$

There are two elements in H_{vex}, and we can enumerate elements in H_{vex} by $\mathbf{H}^i$, where $i = 1,2$,

$$\mathbf{H}^1 = 0, \ \mathbf{H}^1 = h_M. \tag{23}$$

By Lemma 3, $\Phi_{k[h(t)]}$, which is affinely dependent on $h(t) \in H$, can be expressed as follows;

$$\begin{aligned}
\Phi_{k[h(t)]} &= \Phi_{1k} + h(t)\Phi_{2k} \\
&= \Phi_{1k} + [a_1(t)\mathbf{H}^1 + a_2(t)\mathbf{H}^2]\Phi_{2k} \\
&= \Phi_{1k} + [a_1(t) \cdot 0 + a_2(t) \cdot h_M]\Phi_{2k} \\
&= \Phi_{1k} + a_1(t)(0)\Phi_{2k} + a_2(t)(h_M)\Phi_{2k},
\end{aligned} \tag{24}$$

Where; $\Phi_{ik}(i = 1,2)$ are defined in (9). Note that $a_1(t) + a_2(t) = 1$, and $a_i(t) \geq 0 (i = 1,2)$.
Therefore, by using Lemma 3 and convex-hull properties, inequalities (19) hold if the following LMIs are satisfied;

$$\begin{aligned}
\Phi_{k[0]} + XB_k + B_k^T X^T &< 0, \\
\Phi_{k[h_M]} + XB_k + B_k^T X^T &< 0.
\end{aligned} \tag{25}$$

By utilizing Lemma 2 (ii), the inequalities (25) are equivalent to the LMIs (10), respectively, if the LMIs (10) satisfy, then stability condition (18) holds. This completes our proof. ■ As a special case, consider the following CDNs with only time-varying delays in nodes for $i = 1,...,N$;

$$\dot{y}_i(t) = f(y_i(t), y_i(t - h(t))) + c\sum_{j=1}^{N} g_{ij} A y_j(t). \tag{26}$$

Theorem 2: For given scalars $0 < h_M$ and h_D, the network (26) is asymptotically synchronized for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq h_D$, if there exist positive definite matrices $P_k \in \mathbf{R}^{n \times n}$, $Q_{1k} \in \mathbf{R}^{n \times n}$, $Q_{2k} \in \mathbf{R}^{n \times n}$ and $R_k \in \mathbf{R}^{n \times n}$ satisfying the following LMIs for $k = 2,...,N$:

$$\begin{aligned}
(\hat{B}_k^\perp)^T \Phi_{k[0]}(\hat{B}_k^\perp) &< 0, \\
(\hat{B}_k^\perp)^T \Phi_{k[h_M]}(\hat{B}_k^\perp) &< 0,
\end{aligned} \tag{27}$$

Where; $\Phi_{k[h(t)]}$ is defined (9), and;

$$\hat{B}_k = [(J(t) + c\lambda_k A), J_d(t), 0_n, -I_n].$$

Proof: The above criterion is derived in the similar method in the proof of Theorem 1, instead of the matrix B_k, using the matrix $\hat{B}_k$. ■ When the value of the time-derivative of time-delay, $\dot{h}(t)$, is unknown, then, by setting $Q_{2k} = 0$ in (11), we can obtain the following corollaries for the networks (1) and (26), respectively.

Corollary 1: For a given scalar $0 < h_M$, the network (1) is asymptotically synchronized for $0 \leq h(t) \leq h_M$, if there exist positive definite matrices $P_k \in \mathbf{R}^{n \times n}$, $Q_{1k} \in \mathbf{R}^{n \times n}$ and $R_k \in \mathbf{R}^{n \times n}$ satisfying the following LMIs for $k = 2,...,N$:

$$\begin{aligned}
(B_k^\perp)^T \hat{\Phi}_{k[0]}(B_k^\perp) &< 0, \\
(B_k^\perp)^T \hat{\Phi}_{k[h_M]}(B_k^\perp) &< 0,
\end{aligned} \tag{28}$$

Where B_k is defined (9), and;

$$\hat{\Phi}_{k[h(t)]} = \Phi_{k[h(t)]} - \begin{bmatrix} Q_{2k} & 0_n & 0_n & 0_n \\ * & -(1 - h_D)Q_{2k} & 0_n & 0_n \\ * & * & 0_n & 0_n \\ * & * & * & 0_n \end{bmatrix}.$$

Corollary 2: For a given scalar $0 < h_M$, the network (26) is asymptotically synchronized for $0 \leq h(t) \leq h_M$, if there exist positive definite matrices $P_k \in \mathbf{R}^{n \times n}$, $Q_{1k} \in \mathbf{R}^{n \times n}$ and $R_k \in \mathbf{R}^{n \times n}$ satisfying the following LMIs for $k = 2,...,N$:

$$\begin{aligned}
(\hat{B}_k^\perp)^T \hat{\Phi}_{k[0]}(\hat{B}_k^\perp) &< 0, \\
(\hat{B}_k^\perp)^T \hat{\Phi}_{k[h_M]}(\hat{B}_k^\perp) &< 0,
\end{aligned} \tag{29}$$

Where $\hat{B}_k$ and $\hat{\Phi}_{k[h(t)]}$ are defined Theorem 2 and Corollary 1, respectively.

Remark 1: Unlike the work of Li et al. (2008), the presented stability criteria used Finsler's lemma with no free-weighting matrices. Also, the convex properties of $h(t)$ are utilized to derive main results in this work. From this point of view, the stability criteria are more efficient than the one in Li et al. (2008) since it involves the least number of variables while providing less conservative stability conditions. In other words, by employing fewer decision variables, the computation burden and time-consuming will be decreased.

Numerical examples

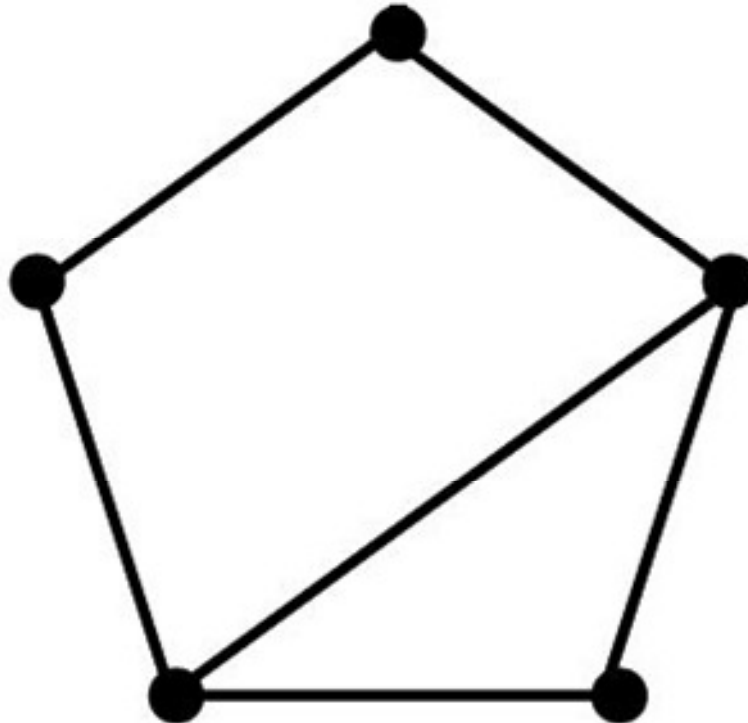In this section, we provide two numerical examples to

Figure 1. The structure of CDNs with five nodes.

show the effectiveness of the presented stability criteria in this paper.

We consider CDNs with 5 nodes (Figure 1) in which each node is an n-order system with the inner-coupling matrix $A = diag\{1,1,1\}$ and the outer-coupling matrix;

$$G = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}.$$

Example 1. Consider the network (1) with the structure in Figure 1 and the following 3-order system;

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \dot{y}_3(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} \quad (30)$$

Which is asymptotically stable at the equilibrium point $s(t) = 0$, and its Jacobian matrices are $J(t) = diag\{-1,-2,-3\}$ and $J_d(t) = 0_n$. The results of the

upper bound of time-delay with different h_D and c provided by theorem 1 and corollary 1 are listed in Table 1. From a computational point of view, the proposed stability criteria are more efficient than the conditions in Li et al. (2008), since it involves the least number of variables while providing the less conservative results. Figure 2 shows the simulation results for the synchronization errors of the network (1) with $h(t) = 1.292\sin^2(0.38t)$ ($h_M = 1.292$, $h_D = 0.5$). This figure shows that the errors between the synchronized states converge to zero under the time-delay $h(t)$.

Example 2. Consider the network (26) with the structure in Figure 1 and the following 3-order system;

$$\dot{y}(t) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & -0.1 \end{bmatrix} y(t) + \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} y(t-h(t)) \quad (31)$$

Which is asymptotically stable at the equilibrium point $s(t) = 0$, and its Jacobian matrices are $J(t) = diag\{-2,0.9,-0.1\}$ and $J_d(t) = \{-1,0,0;1,-1,0;0,0,-1\}$. For the above system, the results of the upper bound of time-delay for different h_D and c are calculated in Table 2. When the value of the time-derivative of time-

delay is unknown, then, by applying Corollary 2 to the above system (31), it can be obtained that the upper

Table 1. The upper bound of time-delay with different h_D and c (example 1).

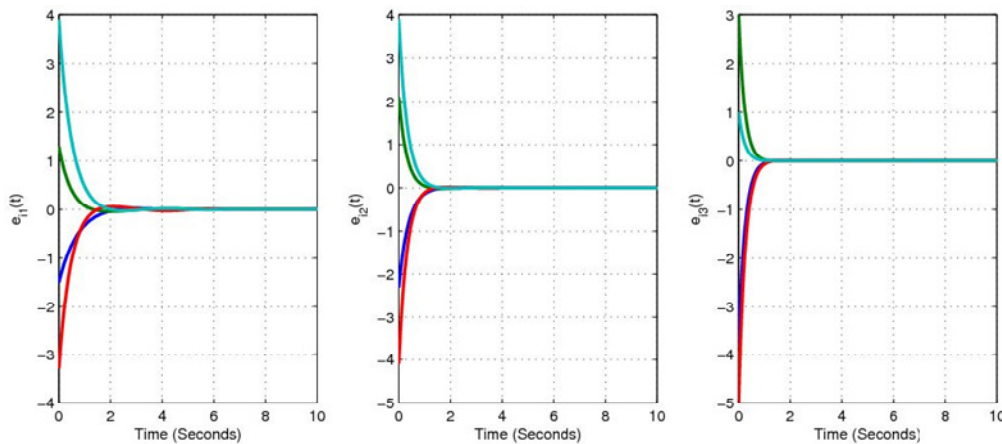| c	h_D	0	0.5	0.9	Unknown	Numbers of variable
	Li et al. (2008)	1.935	1.275	0.970	0.960	444
0.3	Theorem 1	1.935	1.292	1.075	-	96
	Corollary 1	-	-	-	1.075	72
	Li et al. (2008)	1.130	0.854	0.711	0.710	-
0.4	Theorem 1	1.130	0.866	0.780	-	-
	Corollary 1	-	-	-	0.780	-
	Li et al. (2008)	0.813	0.646	0.562	0.562	-
0.5	Theorem 1	0.813	0.655	0.611	-	-
	Corollary 1	-	-	-	0.611	-
	Li et al. (2008)	0.638	0.520	0.464	0.464	-
0.6	Theorem 1	0.638	0.528	0.501	-	-
	Corollary 1	-	-	-	0.501	-

Figure 2. Synchronization errors with $h(t) = 1.292 \sin^2(0.38t)$ (example 1).

bound of time-delay is listed in Table 2. It can also be shown that the proposed stability criteria for system (31) improved the stability region. The simulation results for the synchronization errors of the network (31) with the following conditions are shown in Figures 3 and 4, respectively,

Condition 1: $(c = 0.3, h_M = 1.123, h_D = 0.5)$
$: h(t) = 1.123 \sin^2(0.44t),$

Condition 2: $(c = 0.5, h_M = 1.204$ $h_D : unkwoun) :$

Figures 3 and 4 show that the system (31) with two modes responses converge to zero for given chosen $h(t) = 1.204 |\sin(t)|$. initial values of the state and the above conditions.

CONCLUSIONS

In this paper, new delay-dependent synchronization criteria for the CDNs with time-varying delays are proposed. To obtain less conservative results, the suitable L-K functional is used to improve the feasible region of stability criteria. In addition, they showed that

the presented results contained the least number of variables. Two numerical examples have been given to

Table 2. The upper bound of time-delay with different h_D and c (example 2).

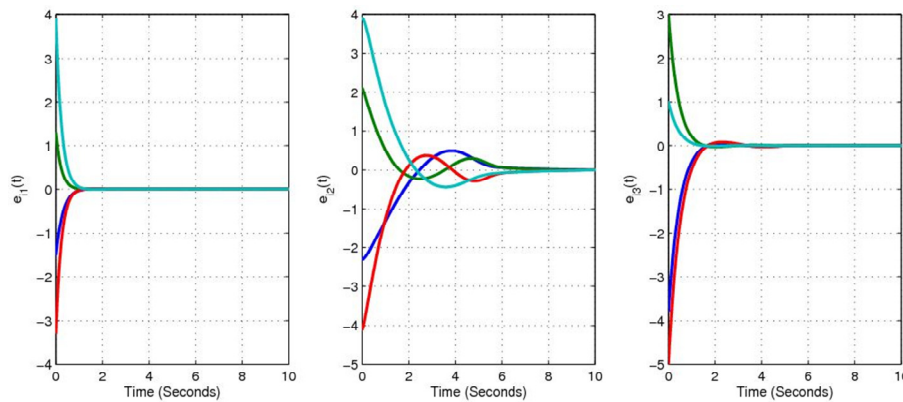| c	h_D	0	0.5	0.9	Unknown
0.3	Theorem 1	1.160	1.123	1.123	-
	Corollary 1	-	-	-	1.123
0.5	Theorem 1	1.286	1.204	1.204	-
	Corollary 1	-	-	-	1.204

Figure 3: Synchronization errors with condition 1 (example 2).
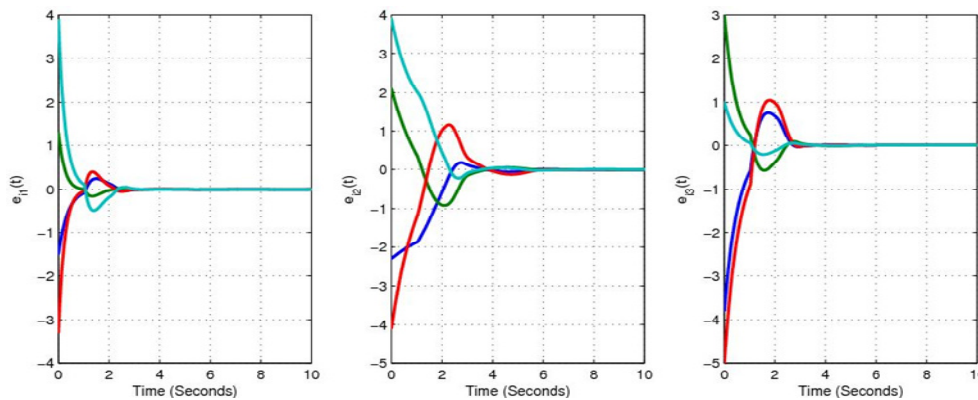

Figure 4: Synchronization errors with conditions 2 (example 2).

show the effectiveness and usefulness of the presented criteria.

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REFERENCES

Boccaletti S, Latora V, Moreno Y, Chavez M, Hwang DU (2006). Complex networks: Structure and dynamics. Phys. Rep., 424: 175.

Boyd S, Ghaoui LE, Feron E, Balakrishnan V (1994). LMI in system and control theory. SIAM, Philadelphia.

de Oliveira MC, Skelton RE (2001). Stability tests for constrained linear systems. Springer-Verlag, Berlin.

Gao H, Lam J, Chen G (2006). New criteria for synchronization of general complex dynamical networks with coupling delays. Phys. Lett. A., 360: 263.

Gu K (2000). An integral inequality in the stability problem of time-delay systems. in: The 39th IEEE Conf. Decision Control, Sydney, Australia, pp. 2805-2810.

Koo JH, Ji DH, Won SC (2010). Synchronization of singular complex dynamical networks with time-varying delays. Appl. Math. Comput., 217: 3916-3923.

Köse IE, Jabbari F, Schmitendorf WE (1998). A Direct Characterization of L_2-Gain Controllers for LPV Systems. IEEE Trans. Autom. Control., 43: 1302-1307.

Kwon OM, Park Ju H, Lee SM (2010). An improved delay-dependent criterion for asymptotic stability of uncertain dynamic systems with time-varying delays. J. Optim. Theory Appl., 145: 343-353.

Li C, Chen G (2004). Synchronization in general complex dynamical networks with coupling delays. Physica A, 343: 263.

Li K, Guan S, Gong X, Lai CH (2008). Synchronization of general complex dynamical networks with time-varying delays. Phys. Lett. A., 372: 7133.

Niculescu SI (2002). Delay effects on stability: a robust approach. Springer-Verlag, New York.

Richard JP (2003). Time-delay systems: an overview of some recent advances and open problems. Automatica, 39: 1667-1694.

Strogatz SH (2001). Exploring complex networks. Nature, 410: 268.

Suplin V, Fridman E, Shaked E (2006). H_∞ control of linear uncertain time-delay systems - a projection approach. IEEE Trans. Autom. Control., 51: 680-685.

Watts DJ, Strogatz SH (1998). Collective dynamics of 'small-world' networks. Nature. 393: 440.

Xu S, Lam J (2007). On Equivalence and Efficiency of Certain Stability Criteria for Time-Delay Systems. IEEE Trans. Autom. Control., 52: 95-101.