Scientific Research and Essays Vol. 7(2), pp. 149-157, 16 January, 2012
Available online at http://www.academicjournals.org/SRE
DOI: 10.5897/SRE11.850
ISSN 1992-2248 ©2012 Academic Journals

Full Length Research Paper

Exact complexiton soliton solutions for nonlinear partial differential equations in mathematical physics

Khaled A. Gepreel[1,2] and A. R. Shehata[3]*

[1]Mathematics Department, Faculty of Science, Taif University, Saudi Arabia.
[2]Mathematics Department, Faculty of Science, Zagazig University, Egypt.
[3]Mathematics Department, Faculty of Science, El-Minia University, Egypt.

Accepted 16 August, 2011

In this article, the extended multiple Riccati equations expansion method has been used to construct a series of double soliton- like solutions, double triangular function solutions and complexiton soliton solutions for some nonlinear partial differential equations in mathematical physics via the (2+1) dimensional breaking soliton equations, (2+1) dimensional painleve integrable Burger's equations and (2+1) dimensional Nizhnik- Novikov- Vesselov equations. With the help of symbolic computation as Maple or Mathematica, we obtain many new types of complexiton soliton solutions, for example, various combination of trigonometric periodic function and hyperbolic function solutions, various combination of trigonometric periodic function and rational function solutions, various combination of hyperbolic function and rational function solutions.

Key words: The extended multiple Riccati equations expansion method, double soliton-like solutions, double triangular function solutions, complexiton soliton solutions, the (2+1) dimensional breaking soliton equations, the (2+1) dimensional painleve integrable Burger's equations, the (2+1)-dimensional Nizhnik-Novikov- Vesselov equations.

INTRODUCTION

In recent years, the exact solutions of non-linear PDEs have been investigated by many authors (Ablowitz and Clarkson, 1991; Miura, 1978; Rogers and Shadwick, 1982; ang and Zhang, 2007; Yan and Zhang, 2001; Liu et al., 2001; Zayed et al., 2004; Abdou, 2007; Fan, 2000; Wang and Li, 2005a, b; He and Wu, 2006; Yusufoglu, 2008; Li and Wang, 2007; Wang et al., 2007; Wang et al., 2005; Zayed et al., 2008a Zhang et al., 2002; Wang et al., 2008; Zhang et al., 2008; He, 2004a, b; Wang et al., 2005; Zayed et al., 2008b; Zhang et al., 2008; Zayed and Gepreel, 2009; Zhang et al., 2008; Zayed and Gepreel, 2011) who are interested in non-linear physical phenomena. Many powerful methods have been presented by those authors such as the inverse scattering transform (Ablowitz and Clarkson, 1991), the Backlund transform (Miura, 1978; Rogers and Shadwick, 1982), the generalized Riccati equation (Wang and Zhang, 2007; Yan and Zhang, 2001), the Jacobi elliptic function expansion (Liu et al., 2001; Zayed et al., 2004), the extended tanh-function method (Abdou, 2007; Fan, 2000), the F-expansion method (Wang and Li, 2005a; b), the exp-function expansion method (He and Wu, 2006; Yusufoglu, 2008), the sub- ODE method (Li and Wang, 2007; Wang et al., 2007), the extended sinh-cosh and sine-cosine methods (Wang et al., 2005), the complex hyperbolic function method (Zayed et al., 2008a), the truncated Painleve expansion (Zhang et al., 2002), the (G´/G)-expansion method (Wang et al., 2008; Zhang et al., 2008), the homotopy perturbation method (He, 2004a, b) and so on. Wang et al. (2005) have presented the multiple Riccati equations rational expansion method, using two or more variables which satisfy different Riccati equations with different parameters. Recently, Zayed et al (2008b) have constructed many new types of complexiton soliton solutions to the generalized coulped nonlinear Hirota-Satsuma equations using Riccati equations method.

*Corresponding author. E-mail: shehata1423@yahoo.com, kagepreel@yahoo.com.

The objective of the present paper is to use the extended multiple Riccati equations expansion method to construct a series of some types of traveling wave solutions of the following nonlinear partial differential equations in mathematical physics:

(i) The (2+1)- dimensional breaking soliton equation (1) (Zhang et al., 2008)

$$u_t + bu_{xxy} + 4bu\,v_x + 4bu_x v = 0,\quad u_y - v_x = 0. \tag{1}$$

(ii) The (2+1)- dimensional painleve integrable Burger's equations (Zayed et al., 2009)

$$-u_t + uu_y + \alpha v u_x + \beta u_{yy} + \alpha\beta u_{xx} = 0,\quad u_x - v_y = 0. \tag{2}$$

(iii) The (2+1)- dimensional Nizhnik-Novikov- Vesselov equations (Zhang et al., 2008)

$$u_t - u_{xxx} - 3(uv)_x = 0,\quad u_x - v_y = 0. \tag{3}$$

Summary of the extended multiple Riccati equations expansion method

In this section, we should like to outline the main steps of this method (Zayed et al., 2011) as follows:

Step 1. We consider the following nonlinear partial differential equation with some physical field $u(x,y,t)$:

$$P(u,u_t,u_x,u_y,u_{tt},u_{xx},u_{xt},u_{yy},...) = 0. \tag{4}$$

Step 2. We introduce a more generalized ansatz in term of a finite formal expansion in the following forms:

$$u(x,y,t) = a_0 + \sum_{k=1}^{n}\sum_{i+j=k} a_i^j \varphi^i(\xi)\psi^j(\eta), \tag{5}$$

where a_0 and $a_i^j\,(i,j=0,1,2,...,n)$ are constants to be determined later, while the new variables $\varphi(\xi)$ and $\psi(\eta)$ satisfy the following two Riccati equations:

$$\varphi'(\xi) = q_1 + r_1\varphi(\xi) + p_1\varphi^2(\xi), \tag{6}$$

and

$$\psi'(\eta) = q_2 + r_2\psi(\eta) + p_2\psi^2(\eta), \tag{7}$$

where q_i, r_i and $p_i, (i=1,2)$ are arbitrary constants such that $p_i \neq 0$. The parameters ξ and η are given by $\xi = k_1x + L_1y + \lambda_1 t$ and $\eta = k_2x + L_2y + \lambda_2 t$, where $k_1,k_2,L_1,L_2,\lambda_1,\lambda_2$ are arbitrary constants.

Step 3. Determine the positive integer "n" of the formal polynomial solution (5) by balancing the highest nonlinear terms and the highest-order partial derivative terms in the given system of the equations, and then we give the formal solution.

Step 4. Substitute from Equation (5) into Equation (4) along with Equation (6) and (7) and then set all the coefficients of $[\varphi(\xi)]^i[\psi(\eta)]^j\,(i,j=0,1,2,...)$ of the resulting system to be zero. We get an over-determined system of nonlinear algebraic equations with respect to $k_1,k_2,L_1,L_2,\lambda_1,\lambda_2,a_0,q_1,q_2,r_1,r_2,p_1,p_2$ and $a_i^j\,(i,j=0,1,2,...,n)$.

Step 5. Solve the over-determined system of nonlinear algebraic equations by using the symbolic computation as Maple or Mathematica. We end up with explicit expressions for $k_1,k_2,L_1,L_2,\lambda_1,\lambda_2,a_0,q_1,q_2,r_1,r_2,p_1,p_2$ and $a_i^j\,(i,j=0,1,2,...,n)$.

Step 6. It is well known that the general solutions of the Riccati equations (6) and (7) are listed as follows:

(i) when $\Delta_1 > 0$ and $\Delta_2 > 0$, then

$$\begin{aligned}
&\varphi(\xi) = -\frac{r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)}{2p_1}, \quad \psi(\eta) = -\frac{r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)}{2p_2},\\
&\varphi(\xi) = -\frac{r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)}{2p_1}, \quad \psi(\eta) = -\frac{r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)}{2p_2},\\
&\varphi(\xi) = -\frac{r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)}{2p_1}, \quad \psi(\eta) = -\frac{r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)}{2p_2},\\
&\varphi(\xi) = -\frac{r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)}{2p_1}, \quad \psi(\eta) = -\frac{r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)}{2p_2},
\end{aligned} \tag{8}$$

(ii) when $\Delta_1 < 0$ and $\Delta_2 < 0$, then;

$$\varphi(\xi)=-\frac{r_1-\sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)}{2p_1},\qquad \psi(\eta)=-\frac{r_2-\sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)}{2p_2},$$
$$\varphi(\xi)=-\frac{r_1-\sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)}{2p_1},\qquad \psi(\eta)=-\frac{r_2+\sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)}{2p_2},\qquad (9)$$
$$\varphi(\xi)=-\frac{r_1+\sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)}{2p_1},\qquad \psi(\eta)=-\frac{r_2-\sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)}{2p_2},$$
$$\varphi(\xi)=-\frac{r_1+\sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)}{2p_1},\qquad \psi(\eta)=-\frac{r_2+\sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)}{2p_2},$$

(iii) when $\Delta_1>0$ and $\Delta_2<0$, the

$$\varphi(\xi)=-\frac{r_1+\sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)}{2p_1},\qquad \psi(\eta)=-\frac{r_2-\sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)}{2p_2},$$
$$\varphi(\xi)=-\frac{r_1+\sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)}{2p_1},\qquad \psi(\eta)=-\frac{r_2+\sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)}{2p_2},\qquad (10)$$
$$\varphi(\xi)=-\frac{r_1+\sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)}{2p_1},\qquad \psi(\eta)=-\frac{r_2-\sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)}{2p_2},$$
$$\varphi(\xi)=-\frac{r_1+\sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)}{2p_1},\qquad \psi(\eta)=-\frac{r_2-\sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)}{2p_2},$$

where $\Delta_1=r_1^2-4p_1q_1$ and $\Delta_2=r_2^2-4p_2q_2$. Similarly we can find other general solutions when $\Delta_1<0$ and $\Delta_2>0$, then which are omitted here for convenience.

(iv) when $r_1=r_2=q_1=q_2=0$ and $p_1,p_2\neq 0$, then

$$\varphi(\xi)=\frac{-1}{p_1\xi+C_1},\qquad \psi(\eta)=\frac{-1}{p_2\eta+C_2},\qquad (11)$$

Where C_1 and C_2 are arbitrary constants**.**

Example 1: The (2+1) dimensional breaking soliton equations

In part of the work, we study the nonlinear (2+1) dimensional breaking soliton Equations (1) using the extended multiple Riccati equations expansion method. By balancing the nonlinear terms and the highest order linear partial derivative terms of (1), we get

$$u(x,y,t)=a_0+a_1\varphi(\xi)+a_2\psi(\eta)+a_3\varphi^2(\xi)+a_4\psi^2(\eta)+a_5\varphi(\xi)\psi(\eta),$$
$$v(x,y,t)=b_0+b_1\varphi(\xi)+b_2\psi(\eta)+b_3\varphi^2(\xi)+b_4\psi^2(\eta)+b_5\varphi(\xi)\psi(\eta),$$
(12)

where $\varphi(\xi)=\varphi(k_1x+L_1y+\lambda_1t)$, $\psi(\eta)=\psi(k_2x+L_2y+\lambda_2t)$ and $k_1,k_2,L_1,L_2,\lambda_1,\lambda_2$, $a_i,b_i,(i=0,1,...,5)$ are constants to be determined later. With the aid of the Maple or Mathematica, we substitute from Equation (12) along with Equations (6) and (7) into Equation (1) and set the coefficients of the terms $[\varphi(\xi)]^i[\psi(\eta)]^j\,(i,j=0,1,2,..)$ to be zero yield. A set of over-determined algebraic equations with respect to $k_1,k_2,L_1,L_2,\lambda_1,\lambda_2$ and $a_i,b_i,(i=0,1,...,5)$. On using the Maple or Mathematica software package, we solve the over-determined algebraic equations. Consequently, we get the following results:

Case 1.

$$a_1=-\frac{3}{2}r_1p_1k_1^2,\ a_2=-\frac{3}{2}r_2p_2k_2^2,\ a_3=-\frac{3}{2}p_1^2k_1^2,\ a_4=-\frac{3}{2}p_2^2k_2^2,$$
$$b_1=\frac{3r_1p_1k_1^2L_2}{2k_2},b_2=-\frac{3}{2}r_2p_2k_2L_2,\ b_3=\frac{3p_1^2k_1^2L_2}{2k_2},L_1=-\frac{k_1L_2}{k_2},\qquad (13)$$
$$b_4=-\frac{3}{2}k_2p_2^2L_2,\ \lambda_2=-8bk_2^2p_2L_2q_2-bk_2^2r_2^2L_2-4L_2a_0b-4bb_0k_2,$$
$$\lambda_1=-\frac{bk_1}{k_2}[-k_1^2r_1^2L_2-8q_1k_1^2p_1L_2+4b_0k_2-4L_2a_0],\ a_5=b_5=0,$$

where $r_1,r_2,q_1,q_2,p_1,p_2,k_2,L_2$ are arbitrary constants.

Case 2.

$$a_2=-\frac{3}{2}r_2p_2k_2^2,\ a_3=-\frac{3}{2}p_1^2k_1^2,\ a_4=-\frac{3}{2}p_2^2k_2^2,\ L_2=-\frac{k_2L_1}{k_1},$$
$$b_2=\frac{3k_2^2r_2p_2L_1}{2k_1},\ b_3=-\frac{3k_1p_1^2L_1}{2},\lambda_1=-4bb_0k_1-4bL_1a_0,\qquad (14)$$
$$b_4=\frac{3k_2^2p_2^2L_1}{2k_1},\ \lambda_2=\frac{bk_2}{k_1}[8k_2^2p_2L_1q_2+k_2^2r_2^2L_1+4L_1a_0-4b_0k_1],$$
$$r_1=q_1=a_5=b_5=a_1=b_1=0,$$

where $r_2,q_2,p_1,p_2,k_2,k_1,L_1$ are arbitrary constants and $p_1,p_2\neq 0$. Note that, there are other cases which are omitted here for convenience. According to (12), (13) and the general solutions (8) - (11) listed in the

Step 6, we obtain the following families of some new types of the double soliton- like solutions, double triangular function solutions and complexiton soliton solutions corresponding to case1, for the nonlinear (2+1) dimensional breaking soliton equations (1):

Family 1. When $\Delta_1 > 0$ and $\Delta_2 > 0$ then the double soliton-like solutions of equation (1.1) have the following forms:

$$\begin{aligned} u_1 &= a_0 + \frac{3}{4} r_1 k_1^2 [r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2^2 [r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)] \\ &\quad - \frac{3}{8} k_1^2 [r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)\ ]^2 - \frac{3}{8} k_2^2 [r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)]^2, \\ v_1 &= b_0 - \frac{3r_1 k_1^2 L_2}{4k_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2 L_2 [r_2 + \sqrt{\Delta_2})\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)] \\ &\quad + \frac{3k_1^2 L_2}{8k_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2 L_2 [r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)]^2. \end{aligned} \tag{16}$$

$$\begin{aligned} u_2 &= a_0 + \frac{3}{4} r_1 k_1^2 [r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2^2 [r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)] \\ &\quad - \frac{3}{8} k_1^2 [r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2^2 [r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)]^2, \\ v_2 &= b_0 - \frac{3r_1 k_1^2 L_2}{4k_2}[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2 L_2 [r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)] \\ &\quad + \frac{3k_1^2 L_2}{8k_2}[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2 L_2 [r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)]^2. \end{aligned} \tag{17}$$

$$\begin{aligned} u_3 &= a_0 + \frac{3}{4} r_1 k_1^2 [r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2^2 [r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)] \\ &\quad - \frac{3}{8} k_1^2 [r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2^2 [r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)]^2, \\ v_3 &= b_0 - \frac{3r_1 k_1^2 L_2}{4k_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2 L_2 [r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)] \\ &\quad + \frac{3k_1^2 L_2}{8k_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2 L_2 [r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)]^2. \end{aligned} \tag{18}$$

$$\begin{aligned} u_4 &= a_0 + \frac{3}{4} r_1 k_1^2 [r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2^2 [r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)] \\ &\quad - \frac{3}{8} k_1^2 [r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2^2 [r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)]^2, \\ v_4 &= b_0 - \frac{3r_1 k_1^2 L_2}{4k_2}[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2 L_2 [r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)] \\ &\quad + \frac{3k_1^2 L_2}{8k_2}[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2 L_2 [r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)]^2. \end{aligned} \tag{19}$$

Family 2. When $\Delta_1 < 0$ and $\Delta_2 < 0$, then the double triangular function solutions of Equation (1.1) have the following forms:

$$\begin{aligned} u_5 &= a_0 + \frac{3}{4} r_1 k_1^2 [r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2^2 [r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &\quad - \frac{3}{8} k_1^2 [r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2^2 [r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2, \\ v_5 &= b_0 - \frac{3r_1 k_1^2 L_2}{4k_2}[r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2 L_2 [r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &\quad + \frac{3k_1^2 L_2}{8k_2}[r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2 L_2 [r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2. \end{aligned} \tag{20}$$

$$\begin{aligned} u_6 &= a_0 + \frac{3}{4} r_1 k_1^2 [r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2^2 [r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &\quad - \frac{3}{8} k_1^2 [r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2^2 [r_2 - \sqrt{\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2, \\ v_6 &= b_0 - \frac{3r_1 k_1^2 L_2}{4k_2}[r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2 L_2 [r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &\quad + \frac{3k_1^2 L_2}{8k_2}[r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2 L_2 [r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2. \end{aligned} \tag{21}$$

$$\begin{aligned} u_7 &= a_0 + \frac{3}{4} r_1 k_1^2 [r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2^2 [r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &\quad - \frac{3}{8} k_1^2 [r_1 + \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2^2 [r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2, \\ v_7 &= b_0 - \frac{3r_1 k_1^2 L_2}{4k_2}[r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2 L_2 [r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &\quad + \frac{3k_1^2 L_2}{8k_2}[r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2 L_2 [r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2. \end{aligned} \tag{22}$$

$$\begin{aligned} u_8 &= a_0 + \frac{3}{4} r_1 k_1^2 [r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2^2 [r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &\quad - \frac{3}{8} k_1^2 [r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2^2 [r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2, \\ v_8 &= b_0 - \frac{3r_1 k_1^2 L_2}{4k_2}[r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2 L_2 [r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &\quad + \frac{3k_1^2 L_2}{8k_2}[r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2 L_2 [r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2. \end{aligned} \tag{23}$$

Family 3. When $\Delta_1 > 0$ and $\Delta_2 < 0$, the complexiton soliton solutions of Equation (1.1) have the following forms:

$$\begin{aligned} u_9 &= a_0 + \frac{3}{4} r_1 k_1^2 [r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2^2 [r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &\quad - \frac{3}{8} k_1^2 [r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2^2 [r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2, \\ v_9 &= b_0 - \frac{3r_1 k_1^2 L_2}{4k_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2 L_2 [r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &\quad + \frac{3k_1^2 L_2}{8k_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2 L_2 [r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta))]^2. \end{aligned} \tag{24}$$

$$\begin{aligned} u_{10} &= a_0 + \frac{3}{4} r_1 k_1^2 [r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2^2 [r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &\quad - \frac{3}{8} k_1^2 [r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2^2 [r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2, \\ v_{10} &= b_0 - \frac{3r_1 k_1^2 L_2}{4k_2}[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2 L_2 [r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &\quad + \frac{3k_1^2 L_2}{8k_2}[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2 L_2 [r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2. \end{aligned} \tag{25}$$

$$\begin{aligned} u_{11} &= a_0 + \frac{3}{4} r_1 k_1^2 [r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2^2 [r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &\quad - \frac{3}{8} k_1^2 [r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2^2 [r_2 + \sqrt{\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2, \\ v_{11} &= b_0 - \frac{3r_1 k_1^2 L_2}{4k_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2 L_2 [r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &\quad + \frac{3k_1^2 L_2}{8k_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2 L_2 [r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2. \end{aligned} \tag{26}$$

$$\begin{aligned} u_{12} &= a_0 + \frac{3}{4} r_1 k_1^2 [r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2^2 [r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &\quad - \frac{3}{8} k_1^2 [r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)\ ]^2 - \frac{3}{8} k_2^2 [r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2, \\ v_{12} &= b_0 - \frac{3r_1 k_1^2 L_2}{4k_2}[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] + \frac{3}{4} r_2 k_2 L_2 [r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &\quad + \frac{3k_1^2 L_2}{8k_2}[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{3}{8} k_2 L_2 [r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2. \end{aligned} \tag{27}$$

Where

$$\xi = k_1 x - \frac{k_1 L_2}{k_2} y - \frac{b k_1 t}{k_2}[-k_1^2 r_1^2 L_2 - 8 q_1 k_1^2 p_1 L_2 + 4 b_0 k_2 - 4 L_2 a_0], \tag{28}$$
$$\eta = k_2 x + L_2 y + (-8 b k_2^2 p_2 L_2 q_2 - b k_2^2 r_2^2 L_2 - 4 L_2 a_0 b - 4 b b_0 k_2) t.$$

According to (12), (14) and the general solutions (8) - (2.8) listed in the step 6, we obtain the following families of some new types of the double soliton- like solutions, double triangular function solutions and complexiton soliton solutions corresponding to case2, for the nonlinear (2+1) dimensional breaking soliton equations (1):

Family 1. When $\Delta_2 > 0$

$$u_{13} = a_0 + \frac{3}{4} r_2 k_2^2 [r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)] - \frac{3 p_1^2 k_1^2}{2(p_1\xi + C_1)^2} - \frac{3}{8} k_2^2 [r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)]^2, \tag{29}$$
$$v_{13} = b_0 - \frac{3 r_2 k_2^2 L_1}{4 k_1}[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)] - \frac{3 p_1^2 k_1 L_1}{2(p_1\xi + C_1)^2} + \frac{3 k_2^2 L_1}{8 k_1}[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)]^2.$$

$$u_{14} = a_0 + \frac{3}{4} r_2 k_2^2 [r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)] - \frac{3 p_1^2 k_1^2}{2(p_1\xi + C_1)^2} - \frac{3}{8} k_2^2 [r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)]^2, \tag{30}$$
$$v_{14} = b_0 - \frac{3 r_2 k_2^2 L_1}{4 k_1}[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)] - \frac{3 p_1^2 k_1 L_1}{2(p_1\xi + C_1)^2} + \frac{3 k_2^2 L_1}{8 k_1}[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)]^2.$$

Family 2. When $\Delta_2 < 0$

$$u_{13} = a_0 + \frac{3}{4} r_2 k_2^2 [r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)] - \frac{3 p_1^2 k_1^2}{2(p_1\xi + C_1)^2} - \frac{3}{8} k_2^2 [r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2, \tag{31}$$
$$v_{13} = b_0 - \frac{3 r_2 k_2^2 L_1}{4 k_1}[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)] - \frac{3 p_1^2 k_1 L_1}{2(p_1\xi + C_1)^2} + \frac{3 k_2^2 L_1}{8 k_1}[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2.$$

$$u_{14} = a_0 + \frac{3}{4} r_2 k_2^2 [r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)] - \frac{3 p_1^2 k_1^2}{2(p_1\xi + C_1)^2} - \frac{3}{8} k_2^2 [r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2, \tag{32}$$
$$v_{14} = b_0 - \frac{3 r_2 k_2^2 L_1}{4 k_1}[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)] - \frac{3 p_1^2 k_1 L_1}{2(p_1\xi + C_1)^2} + \frac{3 k_2^2 L_1}{8 k_1}[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2,$$

where

$$\xi = k_1 x + L_1 y - t[4 b b_0 k_1 + 4 b L_1 a_0], \tag{33}$$
$$\eta = k_2 x - \frac{L_1 k_2}{k_1} y + \frac{b k_2}{k_1}(8 k_2^2 p_2 L_1 q_2 + k_2^2 r_2^2 L_1 + 4 L_1 a_0 - 4 b_0 k_1) t.$$

Example 2: The (2+1) dimensional painleve integrable Burgers equation

In this part of the work, we study the nonlinear (2+1) dimensional painleve integrable Burgers Equation (2) using the extended multiple Riccati equations expansion method. By balancing the nonlinear terms and the highest order linear partial derivative terms of (2), we get

$$u(x,y,t) = a_0 + a_1\varphi(\xi) + a_2\psi(\eta), \tag{34}$$
$$v(x,y,t) = b_0 + b_1\varphi(\xi) + b_2\psi(\eta),$$

where $a_i, b_i, (i = 0,1,2)$ are constants to be determined later. With the aid of the Maple or Mathematica, we substitute (34) along with (6) and (7) into (2) and set the coefficients of the terms $[\varphi(\xi)]^i[\psi(\eta)]^j\,(i, j = 0,1,2,..)$ to be zero yield a set of over-determined algebraic equations with respect to $k_1, k_2, L_1, L_2, \lambda_1, \lambda_2$ and $a_i, b_i, (i = 0,1,2)$. On using the Maple or Mathematica software package, we solve the over-determined algebraic equations. Consequently, we get the following results:

Case 1.

$$a_1 = -2\beta L_1 p_1,\ \ a_2 = -2\beta L_2 p_2,\ b_1 = -\frac{4\beta^2 L_1 L_2 p_1 p_2}{\alpha b_2}, k_1 = \frac{2\beta L_1 L_2 p_2}{\alpha b_2}, \tag{35}$$
$$k_2 = \frac{b_2}{2\beta p_2}, \lambda_1 = \frac{L_1}{\alpha b_2^2}[\beta L_1 r_1 \alpha b_2^2 + a_0 \alpha b_2^2 + 2 b_0 \beta L_2 p_2 \alpha b_2 + 4\beta^3 L_1 L_2^2 p_2^2 r_1],$$
$$\lambda_2 = \frac{1}{4\beta p_2^2}[4\beta^2 L_2^2 p_2^2 r_2 + r_2 \alpha b_2^2 - 2\alpha b_0 b_2 p_2 + 4 a_0 p_2^2 \beta L_2],$$

where $r_1, r_2, q_1, q_2, p_1, p_2, b_2, b_0, L_1, L_2$ are arbitrary constants.

Case 2.

$$a_1 = \frac{\alpha b_1}{\sqrt{-\alpha}},\ \ a_2 = -\sqrt{-\alpha}\, b_2\ \ , k_1 = \frac{\sqrt{-\alpha} L_1}{\alpha}, L_2 = \frac{\sqrt{-\alpha} b_2}{2\beta p_2} \tag{36}$$
$$k_2 = -\frac{b_2}{2\beta p_2},\ \ \lambda_1 = \frac{L_1}{\sqrt{-\alpha}}[a_0\sqrt{-\alpha} - \alpha b_0],\ \lambda_2 = \frac{b_2}{2\beta p_2}[a_0\sqrt{-\alpha} - \alpha b_0],$$

where $p_1, p_2, b_0, b_1, b_2, L_1, L_2$ are arbitrary constants. Note that, there are other cases which are omitted here. According to (34), (35) and the general solutions (8)-(11) listed in the step 6, we obtain the following families of some new types of the double soliton-like solutions, double triangular function solutions and complexiton soliton solutions corresponding to case1, dimensional painleve integrable Burgers Equation (2):

Family 1. When $\Delta_1 > 0$ and $\Delta_2 > 0$ then the double soliton-like solutions of Equation (1.2) have the following forms:

$$u_1 = a_0 + \beta L_1[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] + \beta L_2[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)], \tag{37}$$
$$v_1 = b_0 + \frac{2\beta^2 L_1 L_2 p_2}{\alpha b_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] - \frac{b_2}{2 p_2}[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)].$$

$$u_2 = a_0 + \beta L_1[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] + \beta L_2[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)], \tag{38}$$
$$v_2 = b_0 + \frac{2\beta^2 L_1 L_2 p_2}{\alpha b_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] - \frac{b_2}{2 p_2}[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)].$$

$$u_3 = a_0 + \beta L_1[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] + \beta L_2[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)], \tag{39}$$
$$v_3 = b_0 + \frac{2\beta^2 L_1 L_2 p_2}{\alpha b_2}[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] - \frac{b_2}{2 p_2}[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)].$$

$$u_4 = a_0 + \beta L_1[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] + \beta L_2[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)],$$
$$v_4 = b_0 + \frac{2\beta^2 L_1 L_2 p_2}{\alpha b_2}[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] - \frac{b_2}{2p_2}[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)] \quad (40)$$

Family 2. When $\Delta_1 < 0$ and $\Delta_2 < 0$, then the double soliton- like solutions of Equation (2) have the following forms:

$$u_5 = a_0 + \beta L_1[r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)] + \beta L_2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)],$$
$$v_5 = b_0 + \frac{2\beta^2 L_1 L_2 p_2}{\alpha b_2}[r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)] - \frac{b_2}{2p_2}[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]. \quad (41)$$

$$u_6 = a_0 + \beta L_1[r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)] + \beta L_2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)],$$
$$v_6 = b_0 + \frac{2\beta^2 L_1 L_2 p_2}{\alpha b_2}[r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)] - \frac{b_2}{2p_2}[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]. \quad (42)$$

$$u_7 = a_0 + \beta L_1[r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)] + \beta L_2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)],$$
$$v_7 = b_0 + \frac{2\beta^2 L_1 L_2 p_2}{\alpha b_2}[r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)] - \frac{b_2}{2p_2}[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]. \quad (43)$$

$$u_8 = a_0 + \beta L_1[r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)] + \beta L_2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)],$$
$$v_8 = b_0 + \frac{2\beta^2 L_1 L_2 p_2}{\alpha b_2}[r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)] - \frac{b_2}{2p_2}[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]. \quad (44)$$

Family 3. When $\Delta_1 > 0$ and $\Delta_2 < 0$, the complexiton soliton solutions of Equation (2) have the following forms:

$$u_9 = a_0 + \beta L_1[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] + \beta L_2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)],$$
$$v_9 = b_0 + \frac{2\beta^2 L_1 L_2 p_2}{\alpha b_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] - \frac{b_2}{2p_2}[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]. \quad (45)$$

$$u_{10} = a_0 + \beta L_1[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] + \beta L_2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)],$$
$$v_{10} = b_0 + \frac{2\beta^2 L_1 L_2 p_2}{\alpha b_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] - \frac{b_2}{2p_2}[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]. \quad (46)$$

$$u_{11} = a_0 + \beta L_1[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] + \beta L_2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)],$$
$$v_{11} = b_0 + \frac{2\beta^2 L_1 L_2 p_2}{\alpha b_2}[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] - \frac{b_2}{2p_2}[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]. \quad (47)$$

$$u_{12} = a_0 + \beta L_1[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] + \beta L_2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)],$$
$$v_{12} = b_0 + \frac{2\beta^2 L_1 L_2 p_2}{\alpha b_2}[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] - \frac{b_2}{2p_2}[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]. \quad (48)$$

Where

$$\xi = \frac{2\beta L_1 L_2 p_2}{\alpha b_2}x + L_1 y + \frac{L_1 t}{\alpha b_2^2}[\beta L_1 r_1 \alpha b_2^2 + a_0 \alpha b_2^2 + 2b_0 \beta L_2 p_2 \alpha b_2 + 4\beta^3 L_1 L_2^2 p_2^2 r_1],$$
$$\eta = \frac{b_2}{2\beta p_2}x + L_2 y + \frac{t}{4\beta p_2^2}[4\beta^2 L_2^2 p_2^2 r_2 + r_2 \alpha b_2^2 - 2\alpha b_0 b_2 p_2 + 4a_0 p_2^2 \beta L_2]. \quad (49)$$

Family 4. When $r_1 = q_1 = 0$ and $p_1 \neq 0$, then

(i) When $\Delta_2 > 0$

$$u_{13} = a_0 + \frac{2\beta L_1 p_1}{p_1\xi + C_1} + \beta L_2[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)],$$
$$v_{13} = b_0 + \frac{4\beta^2 L_1 L_2 p_1 p_2}{\alpha b_2(p_1\xi + C_1)} - \frac{b_2}{2p_2}[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)]. \quad (50)$$

$$u_{14} = a_0 + \frac{2\beta L_1 p_1}{p_1\xi + C_1} + \beta L_2[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)],$$
$$v_{14} = b_0 + \frac{4\beta^2 L_1 L_2 p_1 p_2}{\alpha b_2(p_1\xi + C_1)} - \frac{b_2}{2p_2}[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)]. \quad (51)$$

(ii) When $\Delta_2 < 0$

$$u_{15} = a_0 + \frac{2\beta L_1 p_1}{p_1\xi + C_1} + \beta L_2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)],$$
$$v_{15} = b_0 + \frac{4\beta^2 L_1 L_2 p_1 p_2}{\alpha b_2(p_1\xi + C_1)} - \frac{b_2}{2p_2}[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]. \quad (52)$$

$$u_{16} = a_0 + \frac{2\beta L_1 p_1}{p_1\xi + C_1} + \beta L_2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)],$$
$$v_{16} = b_0 + \frac{4\beta^2 L_1 L_2 p_1 p_2}{\alpha b_2(p_1\xi + C_1)} - \frac{b_2}{2p_2}[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]. \quad (53)$$

Where

$$\xi = \frac{2\beta L_1 L_2 p_2}{\alpha b_2}x + L_1 y + \frac{L_1 t}{\alpha b_2^2}[a_0 \alpha b_2^2 + 2b_0 \beta L_2 p_2 \alpha b_2],$$
$$\eta = \frac{b_2}{2\beta p_2}x + L_2 y + \frac{t}{4\beta p_2^2}[4\beta^2 L_2^2 p_2^2 r_2 + r_2 \alpha b_2^2 - 2\alpha b_0 b_2 p_2 + 4a_0 p_2^2 \beta L_2]. \quad (55)$$

Example 3: The (2+1) dimensional Nizhink- Novikov-Vesselvo equation

In this part of the work, we study the nonlinear (2+1) dimensional Nizhink- Novikov-Vesselvo equations (3) using the extended multiple Riccati equations expansion method. By balancing the nonlinear terms and the highest order linear partial derivative terms of (3), we get:

$$u(x,y,t) = a_0 + a_1\varphi(\xi) + a_2\psi(\eta) + a_3\varphi^2(\xi) + a_4\psi^2(\eta) + a_5\varphi(\xi)\psi(\eta),$$
$$v(x,y,t) = b_0 + b_1\varphi(\xi) + b_2\psi(\eta) + b_3\varphi^2(\xi) + b_4\psi^2(\eta) + b_5\varphi(\xi)\psi(\eta) \quad (56)$$

where $a_i, b_i, (i = 0,1,2)$ are constants to be determined later. With the aid of Maple or Mathematica, we substitute from Equation (56) along with (6) and (7) into

Equation (3) and set the coefficients of the terms $[\varphi(\xi)]^i[\psi(\eta)]^j\,(i, j = 0,1,2,..)$ to be zero yield. A set of over-determined algebraic equations with respect to $k_1, k_2, L_1, L_2, \lambda_1, \lambda_2$, $a_i, b_i, (i = 0,1,..5)$. On using the Maple or Mathematica software package, we solve the over-determined algebraic equations. Consequently, we get the following results:

Case 1.

$$a_0 = -\frac{L_2}{3k_2^2}[8k_2^3 p_2 q_2 + k_2^3 r_2^2 - \lambda_2 + 3b_0 k_2],\ a_1 = \frac{2k_1^2 p_1 L_2 r_1}{k_2},\ \ a_2 = -2k_2 p_2 L_2 r_2,$$
$$a_3 = \frac{2k_1^2 p_1^2 L_2}{k_2},\ a_4 = -2k_2 p_2^2 L_2,\ \ b_1 = -2k_1^2 p_1 r_1, b_2 = -2k_2^2 p_2 r_2, b_3 = -2k_1^2 p_1^2,$$
$$b_4 = -2k_2^2 p_2^2, b_5 = a_5 = 0,$$
$$L_1 = -\frac{k_1 L_2}{k_2},\ \lambda_1 = \frac{k_1}{k_2}[8k_1^2 p_1 q_1 k_2 + k_1^2 r_1^2 k_2 + 6b_0 k_2 + 8k_2^3 p_2 q_2 + 8k_2^3 r_2^2 - \lambda_2], \tag{57}$$

Where $r_1, r_2, q_1, q_2, p_1, p_2, b_2, b_0, k_1, L_2, k_2, \lambda_2$ are arbitrary constants. Note that, there are other cases which are omitted here for convenience. According to (56), (57) and the general solutions (8) - (11) listed in the step 6, we obtain the following families of some new types of the double soliton- like solutions, double triangular function solutions and complexiton soliton solutions corresponding to case 1, for the nonlinear (2+1) dimensional Nizhink- Novikov-Vesselvo equation (3).

Family 1. When $\Delta_1 > 0$ and $\Delta_2 > 0$, then the double soliton- like solutions of Equation (1.3) have the following forms:

$$\begin{aligned} u_1 &= -\frac{L_2}{3k_2^2}[8k_2^3 p_2 q_2 + k_2^3 r_2^2 - \lambda_2 + 3b_0 k_2] - \frac{k_1^2 L_2 r_1}{k_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] \\ &+ \frac{k_1^2 L_2}{2k_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 + k_2 L_2 r_2[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)] \\ &- \frac{1}{2}k_2 L_2[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)]^2, \\ v_1 &= b_0 + k_1^2 r_1[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] + k_2^2 r_2[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)] \\ &- \frac{1}{2}k_1^2[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{1}{2}k_2^2[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)]^2. \end{aligned} \tag{58}$$

$$\begin{aligned} u_2 &= -\frac{L_2}{3k_2^2}[8k_2^3 p_2 q_2 + k_2^3 r_2^2 - \lambda_2 + 3b_0 k_2] - \frac{k_1^2 L_2 r_1}{k_2}[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] \\ &+ \frac{k_1^2 L_2}{2k_2}[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 + k_2 L_2 r_2[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)] \\ &- \frac{1}{2}k_2 L_2[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)]^2, \\ v_2 &= b_0 + k_1^2 r_1[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] + k_2^2 r_2[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)] \\ &- \frac{1}{2}k_1^2[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{1}{2}k_2^2[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)]^2. \end{aligned} \tag{59}$$

$$\begin{aligned} u_3 &= -\frac{L_2}{3k_2^2}[8k_2^3 p_2 q_2 + k_2^3 r_2^2 - \lambda_2 + 3b_0 k_2] - \frac{k_1^2 L_2 r_1}{k_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] \\ &+ \frac{k_1^2 L_2}{2k_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 + k_2 L_2 r_2[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)] \\ &- \frac{1}{2}k_2 L_2[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)]^2, \\ v_3 &= b_0 + k_1^2 r_1[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] + k_2^2 r_2[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)] \\ &- \frac{1}{2}k_1^2[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{1}{2}k_2^2[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)]^2. \end{aligned} \tag{60}$$

$$\begin{aligned} u_4 &= -\frac{L_2}{3k_2^2}[8k_2^3 p_2 q_2 + k_2^3 r_2^2 - \lambda_2 + 3b_0 k_2] - \frac{k_1^2 L_2 r_1}{k_2}[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] + \\ &+ \frac{k_1^2 L_2}{2k_2}[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 + k_2 L_2 r_2[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)] \\ &- \frac{1}{2}k_2 L_2[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)]^2, \\ v_4 &= b_0 + k_1^2 r_1[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)] + k_2^2 r_2[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)] \\ &- \frac{1}{2}k_1^2[r_1 + \sqrt{\Delta_1}\coth(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{1}{2}k_2^2[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)]^2. \end{aligned} \tag{61}$$

Family 2. When $\Delta_1 < 0$ and $\Delta_2 < 0$, then the double soliton-like solutions of Equation (1.3) have the following forms:

$$\begin{aligned} u_5 &= -\frac{L_2}{3k_2^2}[8k_2^3 p_2 q_2 + k_2^3 r_2^2 - \lambda_2 + 3b_0 k_2] - \frac{k_1^2 L_2 r_1}{k_2}[r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)] \\ &+ \frac{k_1^2 L_2}{2k_2}[r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)]^2 + k_2 L_2 r_2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &- \frac{1}{2}k_2 L_2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2, \\ v_5 &= b_0 + k_1^2 r_1[r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)] + k_2^2 r_2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &- \frac{1}{2}k_1^2[r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)]^2 - \frac{1}{2}k_2^2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2. \end{aligned} \tag{62}$$

$$\begin{aligned} u_6 &= -\frac{L_2}{3k_2^2}[8k_2^3 p_2 q_2 + k_2^3 r_2^2 - \lambda_2 + 3b_0 k_2] - \frac{k_1^2 L_2 r_1}{k_2}[r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)] + \\ &+ \frac{k_1^2 L_2}{2k_2}[r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)]^2 + k_2 L_2 r_2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &- \frac{1}{2}k_2 L_2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2, \\ v_6 &= b_0 + k_1^2 r_1[r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)] + k_2^2 r_2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &- \frac{1}{2}k_1^2[r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)]^2 - \frac{1}{2}k_2^2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2. \end{aligned} \tag{63}$$

$$\begin{aligned} u_7 &= -\frac{L_2}{3k_2^2}[8k_2^3 p_2 q_2 + k_2^3 r_2^2 - \lambda_2 + 3b_0 k_2] - \frac{k_1^2 L_2 r_1}{k_2}[r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)] + \\ &\frac{k_1^2 L_2}{2k_2}[r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)]^2 + k_2 L_2 r_2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &- \frac{1}{2}k_2 L_2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2, \\ v_7 &= b_0 + k_1^2 r_1[r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)] + k_2^2 r_2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)] \\ &- \frac{1}{2}k_1^2[r_1 - \sqrt{-\Delta_1}\tan(\frac{\sqrt{-\Delta_1}}{2}\xi)]^2 - \frac{1}{2}k_2^2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2. \end{aligned} \tag{64}$$

$$u_8 = -\frac{L_2}{3k_2^2}[8k_2^3 p_2 q_2 + k_2^3 r_2^2 - \lambda_2 + 3b_0 k_2] - \frac{k_1^2 L_2 r_1}{k_2}[r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)] + \frac{k_1^2 L_2}{2k_2}[r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)]^2 + k_2 L_2 r_2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)] - \frac{1}{2}k_2 L_2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2,$$

$$v_8 = b_0 + k_1^2 r_1[r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)] + k_2^2 r_2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)] - \frac{1}{2}k_1^2[r_1 + \sqrt{-\Delta_1}\cot(\frac{\sqrt{-\Delta_1}}{2}\xi)]^2 - \frac{1}{2}k_2^2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2. \quad (65)$$

Family 3. When $\Delta_1 > 0$ and $\Delta_2 < 0$, the complexiton soliton solutions of Equation (1.3) have the following forms:

$$u_9 = -\frac{L_2}{3k_2^2}[8k_2^3 p_2 q_2 + k_2^3 r_2^2 - \lambda_2 + 3b_0 k_2] - \frac{k_1^2 L_2 r_1}{k_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] + \frac{k_1^2 L_2}{2k_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 + k_2 L_2 r_2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)] - \frac{1}{2}k_2 L_2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2,$$

$$v_9 = b_0 + k_1^2 r_1[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] + k_2^2 r_2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)] - \frac{1}{2}k_1^2[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{1}{2}k_2^2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2 \quad (66)$$

$$u_{10} = -\frac{L_2}{3k_2^2}[8k_2^3 p_2 q_2 + k_2^3 r_2^2 - \lambda_2 + 3b_0 k_2] - \frac{k_1^2 L_2 r_1}{k_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] + \frac{k_1^2 L_2}{2k_2}[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 + k_2 L_2 r_2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)] - \frac{1}{2}k_2 L_2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2,$$

$$v_{10} = b_0 + k_1^2 r_1[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)] + k_2^2 r_2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)] - \frac{1}{2}k_1^2[r_1 + \sqrt{\Delta_1}\tanh(\frac{\sqrt{\Delta_1}}{2}\xi)]^2 - \frac{1}{2}k_2^2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2, \quad (67)$$

Where

$$\xi = k_1 x - \frac{k_1 L_2}{k_2}y + \frac{k_1 t}{k_2}[8k_1^2 p_1 q_1 k_2 + k_1^2 r_1^2 k_2 + 6b_0 k_2 + 8k_2^3 p_2 q_2 + 8k_2^3 r_2^2 - \lambda_2],$$
$$\eta = k_2 x + L_2 y + \lambda_2 y. \quad (68)$$

Family 4. If $r_1 = q_1 = 0$ and $p_1 \neq 0$ then

(i) When $\Delta_2 > 0$

$$u_{11} = -\frac{L_2}{3k_2^2}[8k_2^3 p_2 q_2 + k_2^3 r_2^2 - \lambda_2 + 3b_0 k_2] - \frac{2k_1^2 L_2 r_1 p_1}{k_2(p_1\xi + C_1)} + \frac{2k_1^2 L_2 p_1^2}{k_2(p_1\xi + C_1)^2} + k_2 L_2 r_2[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)] - \frac{1}{2}k_2 L_2[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)]^2, \quad (69)$$

$$v_{11} = b_0 + \frac{2p_1 k_1^2 r_1}{(p_1\xi + C_1)} + k_2^2 r_2[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)] - \frac{2k_1^2 p_1^2}{(p_1\xi + C_1)^2} - \frac{1}{2}k_2^2[r_2 + \sqrt{\Delta_2}\tanh(\frac{\sqrt{\Delta_2}}{2}\eta)]^2.$$

$$u_{12} = -\frac{L_2}{3k_2^2}[8k_2^3 p_2 q_2 + k_2^3 r_2^2 - \lambda_2 + 3b_0 k_2] - \frac{2k_1^2 L_2 r_1 p_1}{k_2(p_1\xi + C_1)} + \frac{2k_1^2 L_2 p_1^2}{k_2(p_1\xi + C_1)^2} + k_2 L_2 r_2[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)] - \frac{1}{2}k_2 L_2[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)]^2,$$

$$v_{12} = b_0 + \frac{2p_1 k_1^2 r_1}{(p_1\xi + C_1)} + k_2^2 r_2[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)] - \frac{2k_1^2 p_1^2}{(p_1\xi + C_1)^2} - \frac{1}{2}k_2^2[r_2 + \sqrt{\Delta_2}\coth(\frac{\sqrt{\Delta_2}}{2}\eta)]^2. \quad (70)$$

(ii) When $\Delta_2 < 0$

$$u_{13} = -\frac{L_2}{3k_2^2}[8k_2^3 p_2 q_2 + k_2^3 r_2^2 - \lambda_2 + 3b_0 k_2] - \frac{2k_1^2 L_2 r_1 p_1}{k_2(p_1\xi + C_1)} + \frac{2k_1^2 L_2 p_1^2}{k_2(p_1\xi + C_1)^2} + k_2 L_2 r_2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)] - \frac{1}{2}k_2 L_2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2,$$

$$v_{13} = b_0 + \frac{2p_1 k_1^2 r_1}{(p_1\xi + C_1)} + k_2^2 r_2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)] - \frac{2k_1^2 p_1^2}{(p_1\xi + C_1)^2} - \frac{1}{2}k_2^2[r_2 - \sqrt{-\Delta_2}\tan(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2. \quad (71)$$

$$u_{14} = -\frac{L_2}{3k_2^2}[8k_2^3 p_2 q_2 + k_2^3 r_2^2 - \lambda_2 + 3b_0 k_2] - \frac{2k_1^2 L_2 r_1 p_1}{k_2(p_1\xi + C_1)} + \frac{2k_1^2 L_2 p_1^2}{k_2(p_1\xi + C_1)^2} + k_2 L_2 r_2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)] - \frac{1}{2}k_2 L_2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2,$$

$$v_{14} = b_0 + \frac{2p_1 k_1^2 r_1}{(p_1\xi + C_1)} + k_2^2 r_2[r_2 - \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)] - \frac{2k_1^2 p_1^2}{(p_1\xi + C_1)^2} - \frac{1}{2}k_2^2[r_2 + \sqrt{-\Delta_2}\cot(\frac{\sqrt{-\Delta_2}}{2}\eta)]^2. \quad (72)$$

where

$$\xi = k_1 x - \frac{k_1 L_2}{k_2}y + \frac{k_1 t}{k_2}[6b_0 k_2 + 8k_2^3 p_2 q_2 + 8k_2^3 r_2^2 - \lambda_2], \eta = k_2 x + L_2 y + \lambda_2 y. \quad (73)$$

Conclusion

In summary, the extended multiple Riccati equations expansion method with symbolic computation is developed to the nonlinear partial differential equations in mathematical physics via the (2+1) dimensional breaking soliton equations, (2+1) dimensional painleve integrable Burgers equation and (2+1)-dimensional Nizhnik-Novikov- Vesselov equation. Then, when applying the proposed method to these Equations (1.1), (1.2) and (1.3), a rich variety of exact solutions which include (a) double solitary- like wave solutions, (b) double trigonometric function solutions, (c) complexiton soliton solutions, are obtained.

REFERENCES

Abdou MA (2007). The extended tanh-method and its applications for solving nonlinear physical models. Appl. Math. Comput., 190 : 988-996.

Ablowitz MJ, Clarkson PA (1991). Solitons, non-linear Evolution Equations and Inverse Scattering Transform. Cambridge Univ. Press, Cambridge .

Fan EG (2000). Extended tanh- function method and its applications to nonlinear equations. Phys. Lett., A. 277: 212-218.

He JH (2004). Comparison of homotopy perturbation method and homotopy analysis method. Appl. Math. Comput., 156: 527-539.

He JH (2004). The homotopy perturbation method for nonlinear oscillators with discontinuities. Appl. Math. Comput., 151: 287-292.

He JH, Wu XH (2006). Exp-function method for nonlinear wave equations. Chaos Solit. Fract., 30: 700-708.

Li XZ, Wang ML (2007). A sub-ODE method for finding exact solutions of a generalized KdV-mKdV equation with higher-order nonlinear terms. Phys. Lett. A., 361: 115-118.

Liu S, Fu Z, Liu SD, Zhao Q (2001). Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. Phys. Lett., A. 289: 69-74.

Miura MR (1978). Backlund Transformation.Springer-Verlag, Berlin

Rogers C, Shadwick WF (1982). Backlund Transformations. Academic Press, New York .

Wang DS, Ren YJ, Zhang HQ (2005). Further extended sinh- cosh and sin-cos methods and new non- travelling wave solutions of the (2+1)-dimensional dispersive long wave equations. Appl. Math. E-Notes., 5: 157-163.

Wang M, Li X (2005). Extended F-expansion and periodic wave solutions for the generalized Zakharov equations. Phys. Lett. A., 343: 48-54.

Wang M, Li X (2005). Applications of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equation. Chaos Solit. Fract., 24: 1257- 1268.

Wang M, Li X, Zhang J (2008). The (G´/G) - expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. Phys. Lett. A., 372: 417-423.

Wang ML, Li XZ , Zhang JL (2007). Sub-ODE method and solitary wave solutions for higher-order nonlinear Schrodinger equation. Phys. Lett. A., 363 : 96-101.

Wang Q, Chen Y, Zhang H (2005). A new Riccati equation rational expansion method and its application to (2+1) - dimensional Burgers equation, Chaos Solit. Fract., 25: 1019-1028.

Wang Z, Zhang HQ (2007). A new generalized Riccati equation rational expansion method to a class of nonlinear evolution equation with nonlinear terms of any order. Appl. Math. Comput., 186 : 693-704.

Yan Z, Zhang H (2001). New explicit solitary wave solutions and periodic wave solutions for Whitham-Broer-Kaup equation in shallow water. Phys. Lett. A., 285: 355-362.

Yusufoglu E (2008). New solitary solutions for the MBBM equations using Exp-function method. Phys. Lett. A., 372: 442-446.

Zayed EME, Zedan HA, Gepreel KA (2004). On the solitary wave solutions for nonlinear Hirota-Satsuma coupled KdV equations. Chaos Solit. Fract., 22: 285-303.

Zayed EME, Gepreel KA (2011). A series of complexiton soliton solutions for nonlinear Jaulent- Miodek PDEs using Riccati equations method, Proc. Royal Soc. Edinb. Sect. A Math., 181: 1001-1015.

Zayed EME, Gepreel KA(2009). The generalized multiple Riccati equations rational expansion method and its application to the (2+1) –dimensional painleve integrable Burgers equation. Int. J. Nonlin. Sci. Num. Simul., 10: 839-849.

Zayed EME, Gepreel KA, Horbaty MM (2008). Exact solutions for some nonlinear differential equations using complex hyperbolic function. Appl. Anal., 87: 509-522.

Zayed EME, Nofal TA, Gepreel KA (2008). A series of complexiton soliton solutions for nonlinear Hirota-Satsuma equations using the generalized multiple Riccati equations rational expansion method. Appl. Anal., 87: 987-1003.

Zhang J, Wei X, Lu Y(2008). A generalized (G'/G)-expansion method and its applications. Phys. Lett. A., 372: 3653-3658.

Zhang S, Tong J, Wang W (2008). A generalized (G´/G) - expansion method for the mKdV equation with variable coefficients. Phys. Lett. A., 372: 2254-2257.

Zhang S, Zheng F, Yu J, Ji M, Lu Z, Ma C (2008). A Generalized F-expansion method and its application to (2+1) dimensional Breaking soliton equations, Int. J. Nonlin. Sci., 5: 25-32.

Zhang SL, Wu B, Lou SY (2002). Painleve analysis and special solutions of generalized Broer-Kaup equations. Phys. Lett. A., 300: 40-48.