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Linear thermoelastic analysis of a functionally graded (FG) rotating disk with different boundary conditions using Adomian's decomposition method

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Linear thermoelastic formulation of a functionally graded rotating disk is studied in the present paper. Thermoelastic solution of a rotating disk, due to the nonhomogenous terms, needs more mathematic operation and the solution of the governing equation is more complicated. In the present paper, thermo elastic formulation of a functionally graded rotating disk is presented and Adomian's decomposition method is employed to solve the obtained equation. Two types of boundary condition are studied including annular disk with two free stress and two fixed edges. Results of the stress and the displacement are investigated for two types of boundary conditions, individually. Results of the present paper can be applicable in the analysis of the rotational components. The numerical results indicate that the radial distribution of the radial and hoop stress for all values of nonhomogenous index have an intersection point at an identical radial coordinate.

Key words: Rotating disk, functionally graded material, Adomian's decomposition method, solution.

INTRODUCTION

Functionally graded materials for the first time was created in laboratory. Properties of these materials can vary continuously and gradually along the components of coordinate system. This varying can be simulated using an analytical function. Main application of functionally graded materials (FGMs) is in the environment with high temperature gradient. Chemical and nuclear reactors, turbine blades and spacecrafts are the known examples for described environments. FGMs are a mixture of ceramic and a ductile metal. Ceramics have high modulus of elasticity that can be used as stiff material. Metals have a ductility property that causes FGMs to be deformable. Ceramics keep their configuration under high temperature environment. Because of natural property of ceramics, at high temperature, it reaches to a brittle state. This brittle state can be modified by compounding it with metals. Therefore, FGMs have a ductility property with scientist, have created new class of materials. Properties of this material varied continuously in terms of constant configuration in high temperature environments. Primarily in 1984, one Japanese group of material scientist have created new class of materials. Properties of this material varied continuously in terms of components of coordinate system. In the following, in the first years of decade 1990, it is started research on the thermal and vibration analysis of FG materials (Yamanouchi et al., 1990).

Tutuncu and Ozturk (2001) presented the exact solution of spherical and cylindrical pressure vessel made of FGM. By applying the internal pressure on the inner radius of vessel, they obtained distribution of the displacement and stress. Jabbari et al. (2002) analyzed the thermo elastic analysis of a FG cylinder under thermal and mechanical loads. They supposed that the material properties vary as a power function in terms of

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the radial coordinate system. After substitution of the temperature distribution in the navier equation, the obtained differential equation was solved, analytically.

Hosseini and Naghdabadi (2007) studied thermoelastic analysis of a rotating disk made of functionally graded materials. They used equilibrium equation by regarding the stresses and strains in thermal environment and centrifugal body force. Final derived governing equation of a FG rotating disk that was presented in that paper seems to be wrong. In this paper, true governing differential equation of equilibrium is presented, exactly. For this reason, the present paper does not include any comparison between obtained results in this paper and that obtained results in literature (Hosseini and Naghdabadi, 2007).

Alinia and Ghannadpour (2009) studied non-linear analysis of square plate under pressure. First order shear deformation theory (FSDT) was used for the evaluation of displacement in the coordinate system. The geometric nonlinearity was considered in the relation of strain-displacement. In this paper, it was guessed displacement such that satisfies the homogenous boundary conditions of plate. By minimizing of energy with respect to amplitude of assumed displacements, complete solution of the displacement and then strains and stresses have been presented with arbitrary precision.

Hojjati and Safari (2008) studied the elastic solution of a rotating disk with nonuniform distribution of thickness and density. They used the homotopy perturbation and Adomian's decomposition method for solving the governing differential equation. These two methods are the adequate tools for solving the non-homogenous linear and nonlinear differential equations.

Allahverdizadeh et al. (2008) studied nonlinear and dynamic analyses of a circular plate. They employed equilibrium and compatibility equations. With introduction of airy function, a system of nonlinear-coupled differential equation was derived. By employing a simple changing of variable, they eliminated time from the obtained differential equations.

Bayat et al. (2009) present a theoretical solution for thermoelastic analysis of functionally graded (FG) rotating disk with variable thickness based on first-order shear deformation theory (FSDT). It was difficult to propose an analytical approach for solving the derived differential equations. Therefore, they used layer wise theory for solving the problem.

Layer wise theory was used for analyzing functionally graded composite plates in cylindrical bending that is subjected to thermo mechanical loadings by Tahani and Mirzababaee (2009). The nonlinear strain–displacement relations were used to study the effect of geometric nonlinearity. The equilibrium equations were solved by using the perturbation technique. Theoretical analysis for the FG thin plates based on the physical neutral surface was studied by Zhang and Zhou (2008). some related works about functionally graded and functionally graded piezoelectric materials performed by the authors of this paper (Rahimi et al., 2011; Arefi et al., 2011; Arefi and Rahimi, 2011, 2012).

In the present paper, it is the main objective to derive the exact thermoelastic formulation of a FG rotating disk. Based on knowledge of authors, presented formulation in the literature (Hosseini et al., 2007) is wrong. Because of this incompleteness, we focus on the true derivation of the formulation of a FG rotating disk. In the following, it is convenient to present the results of new formulation for two applicable types of boundary conditions.

FGM properties

In this part of paper, the properties of functionally graded materials are considered. As mentioned in introduction, FGMs have properties that vary in terms of components of coordinate system. The important property in which many paper is studied, is modulus of elasticity. In this paper, properties such as density, heat expansion and heat conduction coefficients suppose to vary as a power function. Proposed function for variable coefficient can be presented as follows:

$E(r) = E_0 r^n$ Modulus of elasticity

$\rho(r) = \rho_0 r^n$ Density

$k(r) = k_0 r^n$ Heat conduction coefficient

$\alpha(r) = \alpha_0 r^n$ Heat expansion coefficient

Numerical values of $E_0, k_0, \alpha_0, \rho_0$ can be supposed for every problem, individually.

FORMULATION

In this part, basic relation for analysis of a FG rotating disk is presented. These equations were derived in the temperature environment. Because of small ratio of thickness respect to other dimensions of structure, this model can be analyzed by plane stress state. It is supposed that the model to be axisymmetric and consequently only nonzero component of displacement is radial displacement. Based on supposed condition, the radial and circumferential strains can be expressed using Equation (1):

$$\varepsilon_r = \frac{du}{dr}, \ \varepsilon_\theta = \frac{u}{r} \tag{1}$$

Stress strain relation in the plane stress state and under temperature gradient can be obtained as follows:

$$\varepsilon_r = \frac{\sigma_r - v\,\sigma_\theta}{E(r)} + \alpha T$$
$$\varepsilon_\theta = \frac{\sigma_\theta - v\,\sigma_r}{E(r)} + \alpha T \tag{2}$$

Therefore, components of stress can be derived from Equation (3):

$$\sigma_r = \frac{E(r)}{1-\nu^2}[(\varepsilon_r + \nu\,\varepsilon_\theta) - (1+\nu)\alpha T]$$
$$\sigma_\theta = \frac{E(r)}{1-\nu^2}[(\varepsilon_\theta + \nu\,\varepsilon_r) - (1+\nu)\alpha T] \quad (3)$$

The equilibrium equation in presence of centrifugal force is:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \rho(r) r\omega^2 = 0 \quad (4)$$

Substituting of radial and circumferential stresses from Equation (3) into Equation (4), yields final differential equation in terms of the radial displacement.

$$r\frac{dE}{dr}\frac{1}{1-\nu^2}[\frac{du}{dr} + \nu \times \frac{u}{r} - (1+\nu)\alpha T] + \frac{rE}{1-\nu^2}[\frac{d^2u}{dr^2} + \nu\{\frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\} - (1+\nu)\frac{d(\alpha T)}{dr}]$$
$$+\frac{E}{1-\nu^2}[\frac{du}{dr} + \nu\frac{u}{r} - (1+\nu)\alpha T] - \frac{E}{1-\nu^2}[\nu\frac{du}{dr} + \frac{u}{r} - (1+\nu)\alpha T] + \rho r^2\omega^2 = 0 \quad (5)$$

With arrangement of variable, *u* in a decreasing manner, final governing differential equation can be expressed as follows:

$$\frac{d^2u}{dr^2} + [\frac{1}{E}\frac{dE}{dr} + \frac{1}{r}]\frac{du}{dr} + [\frac{\nu}{rE}\frac{dE}{dr} - \frac{1}{r^2}]u - \frac{1}{E}\frac{dE}{dr}(1+\nu)\alpha T - (1+\nu)\frac{d(\alpha T)}{dr}$$
$$+\frac{(1-\nu^2)\rho r\omega^2}{E} = 0 \quad (6)$$

Equation (6) is the final governing differential equation of a rotating disk in thermal environment. Details of procedure of obtaining Equation (6) can be considered in the Appendix 1. In this equation, three non homogenous terms exist. Different writers applied numerical and semi exact methods for evaluating of the solution. Applied methods are very long and are applicable for a special geometry and properties. These methods spend so much time for evaluating the solution for a special case. Therefore, in the present paper, Adomian's decomposition method is proposed as an analytical solution for solving the obtained differential equation with the general conditions.

For evaluation of the solution of a typical nonlinear differential equation, in the first step, the differential equation is divided to numerous terms that are described as follows:

$$Lu + Ru + Nu = g$$

L: Highest order operator of differential equation

R: Lower order linear operator

N: Nonlinear operator of differential equation

g: Nonhomogenous term of differential equation

$$L(u) + R(u) + N(u) = f(r)$$
$$\begin{cases} L(u) = \dfrac{d^2(...)}{dr^2} \\ R(u) = [\dfrac{1}{E(r)}\dfrac{dE(r)}{dr} + \dfrac{1}{r}]\dfrac{d(...)}{dr} + [\dfrac{\nu}{rE(r)}\dfrac{dE}{dr} - \dfrac{1}{r^2}](...) \\ N(u) = 0 \\ f(r) = \dfrac{(1+\nu)}{E(r)}[\dfrac{dE}{dr}]\alpha T + (1+\nu)\dfrac{d(\alpha T)}{dr} - \dfrac{(1-\nu^2)}{E(r)}\rho r\omega^2 \end{cases}$$

Here, we consider a power function for modulus of elasticity (Arefi and Rahimi, 2010). Based on this assumption, we can reproduce the defined operator in Equation (7) as follows:

$$E(r) = E_0 r^n \rightarrow$$
$$\begin{cases} L(u) = \dfrac{d^2(...)}{dr^2} \\ R(u) = [\dfrac{n}{r} + \dfrac{1}{r}]\dfrac{d(...)}{dr} + [\dfrac{\nu n}{r^2} - \dfrac{1}{r^2}](...) \\ N(u) = 0 \\ f(r) = \dfrac{(1+\nu)n}{r}\alpha T + (1+\nu)\dfrac{d(\alpha T)}{dr} - \dfrac{(1-\nu^2)}{E_0}\rho r^{1-n}\omega^2 \end{cases} \quad (8)$$

Presented relations in Equation (8), are the final relations that must be applied in Adomians decomposition method (ADM).

Derivation of temperature distribution along the radial coordinate system is first necessary step in solution of the problem.

Distribution of temperature

Here it is supposed that the temperature of inner and outer layers to be *Ta* and *Tb*, respectively. Regarding the cylindrical coordinate system and symmetric state, the governing differential equation of heat transfer is:

$$\frac{\partial}{\partial r}[rk(r)\frac{\partial T(r)}{\partial r}] = 0 \quad (9)$$

As stated in FG material, heat conduction coefficient must be expresses as a power function of radial displacement, as follows:

$$k = k_0 r^n \quad (10)$$

With regardness to the previous-proposed function, solution of heat conduction is obtained as follows:

$$T(r) = c1 + c2 \times r^{-n} \quad (11)$$

By imposing the assumed boundary conditions to Equation (11), distribution of temperature can be expressed using Equation (12):

$$\begin{cases} T(r=a) = T_a \\ T(r=b) = T_b \end{cases}$$
$$T(r) = \frac{T_a a^n - T_b b^n}{a^n - b^n} - \frac{T_a - T_b}{-a^{-n} + b^{-n}} r^{-n} \quad (12)$$

Finding the homogenous and non-homogenous solution as Zeroth order solution

Zeroth order solution of assumed differential equation include two parts: first part is equal to homogenous solution of differential equation. Second part of solution is obtained by considering the non-homogenous. Homogenous solution of the differential Equation (6) can be obtained by regarding the left side of differential equation as follows:

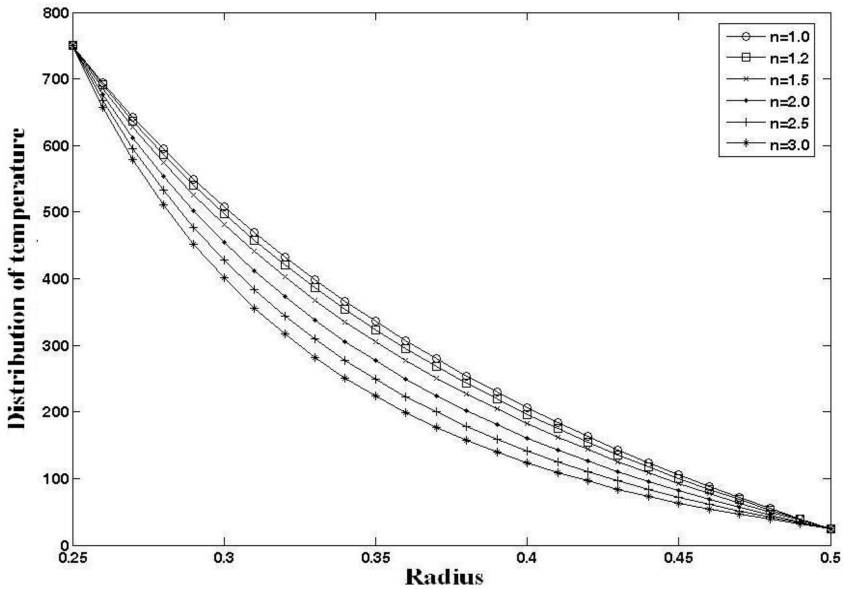

Figure 1. The radial distribution of the temperature in terms of different value of non-homogenous coefficient.

$$\frac{d^2u}{dr^2}+[\frac{n}{r}+\frac{1}{r}]\frac{du}{dr}+[\frac{\nu n}{r^2}-\frac{1}{r^2}]u=0$$

$$u_h=c_1r^{-\frac{n}{2}+\frac{\sqrt{n^2-4\nu n+4}}{2}}+c_2r^{-\frac{n}{2}-\frac{\sqrt{n^2-4\nu n+4}}{2}} \quad (13)$$

Right side of Equation (6) is:

$$f(r)=\frac{(1+\nu)n}{r}\alpha T+(1+\nu)\frac{d(\alpha T)}{dr}-\frac{(1-\nu^2)}{E_0}\rho r^{1-n}\omega^2 \quad (14)$$

Non homogenous solution of zeroth order can be obtained by two times integration with respect to r (because that $L(u)=\frac{d^2(...)}{dr^2}$, inverse operator of L can be shown by $L^{-1}=\int\int(...)drdr$).

$$u_0=u_h+L^{-1}[f(r)]=c_1r^{-\frac{n}{2}+\frac{\sqrt{n^2-4\nu n+4}}{2}}+c_2r^{-\frac{n}{2}-\frac{\sqrt{n^2-4\nu n+4}}{2}}+L^{-1}[f(r)]$$
$$L^{-1}=\int\int(...)drdr \quad (15)$$

Finding the higher order solution

Having the 0th order solution, higher order solutions such as 1st and 2nd solution can be obtained by the reversible Equation (16) as follows:

$$u_n=-L^{-1}[\frac{1+n}{r}\frac{\partial u_{n-1}}{\partial r}]-L^{-1}[\frac{\nu n-1}{r^2}u_{n-1}] \qquad n\geq 1 \quad (16)$$

Final solution is summation of the obtained solutions from 0th order to achievable obtained order. Order of solution depends on the author or software ability for doing the integration and convergence of solution. Therefore, final solution is equal to:

$$u=\sum_{n=0}^{l}u_n \quad (17)$$

As shown in Equation (15), solution includes two constants c1, c2 and consequently, final solution would include two prescribed constants. After obtaining final solution, assumed boundary conditions must be imposed on Equation (17).

RESULTS

The radial distribution of temperature can be obtained for different values of non-homogenous index, *n* by imposing the necessary boundary conditions $T_a=750, T_b=25$. Figure 1 shows the radial distribution of temperature for different values of *n*. Inner and outer radii are supposed 0.25 and 0.5, respectively. By substitution of temperature rising, the other distribution can be obtained. This figure indicates that the value of temperature decreases with increasing of the non-homogenous index.

The other numerical values are considered as follows:

$$E_0=\frac{2e11}{b^n} \qquad k_0=\frac{500}{b^n} \qquad \alpha_0=\frac{5e^{-6}}{b^n} \qquad \rho_0=\frac{7800}{b^n}$$

Investigating on the two different boundary conditions

Fixed support at inner and outer radii

As a first instance, it is supposed that displacement

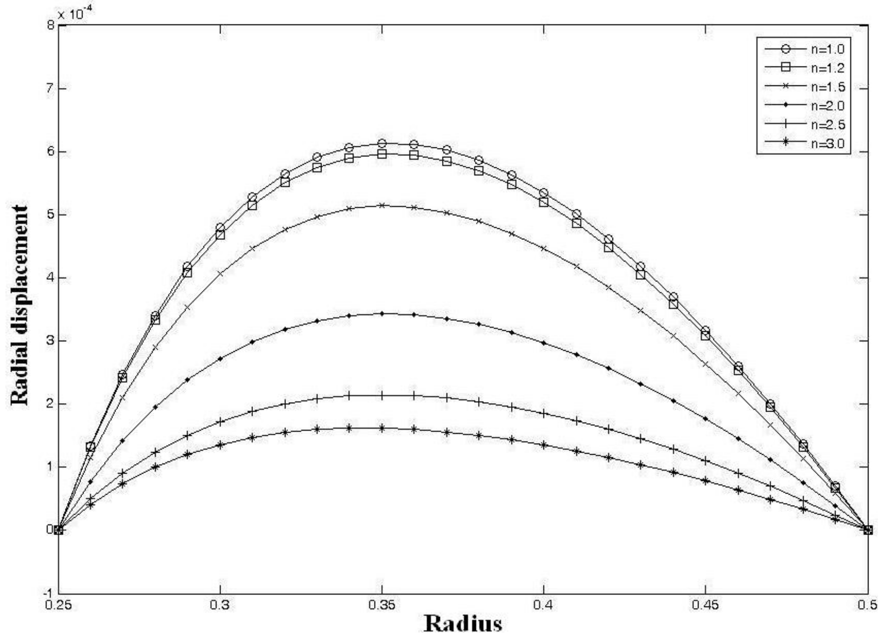

Figure 2. The radial displacement of a disk for different value of n.

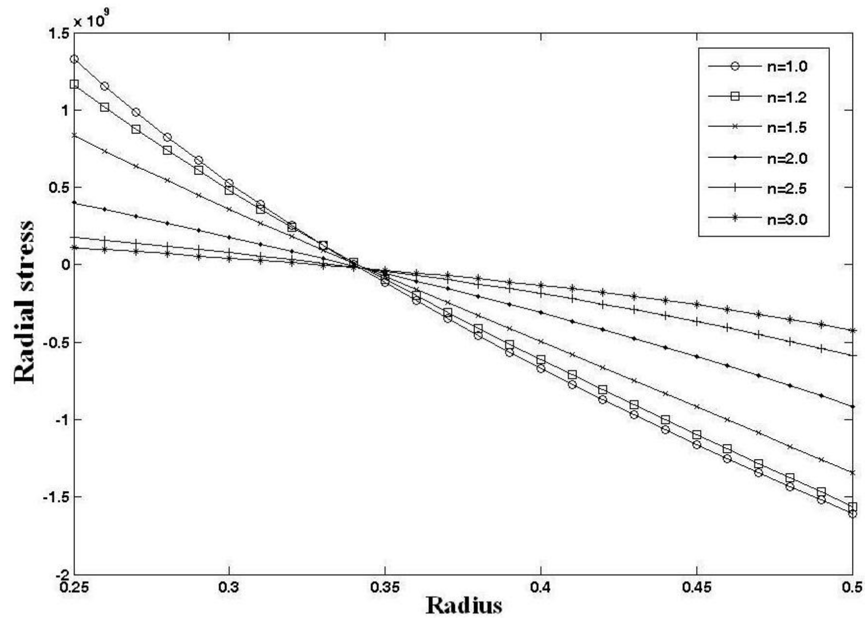

Figure 3. The radial stress of a disk with the fixed inner and outer radius for different value of n.

vanishes at inner and outer radii.

$$B.C\begin{cases}u(r=a)=0\\u(r=b)=0\end{cases} \tag{18}$$

Figure 2 shows the radial displacement of a rotating disk under thermal environment and assumed boundary condition (Equation 18) in terms of non-homogenous index (*n*). As shown in this figure, the radial displacement decreases with increasing the non-homogenous index. This result can be interpreted and justified by using this fact that the radial rigidity of disk increases with increasing the non-homogenous index and consequently, the radial displacement decreases.

Figures 3 and 4 show the radial and circumferential

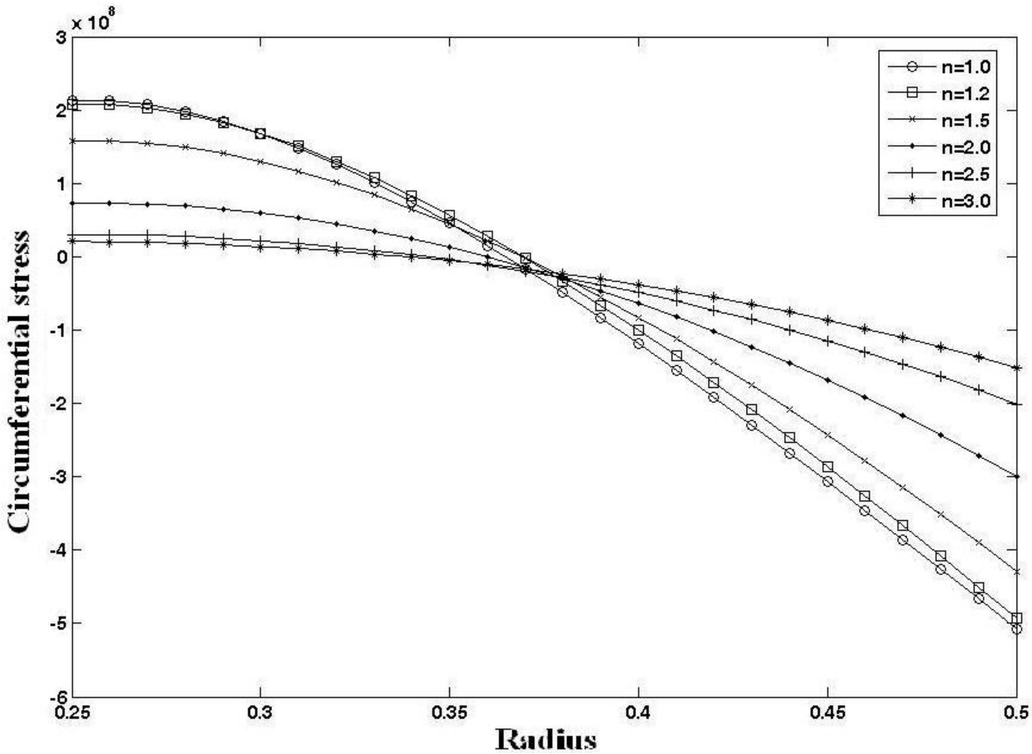

Figure 4. The circumferential stress of disk with fixed inner and outer radii for different value of n.

stress for different values of (*n*), respectively. The distribution of the radial stress in terms of different non-homogenous index depends on the radial coordinate. Different distributions corresponding to different values of non-homogenous index have an intersection point at r=0.34. For the radial coordinate lower than this value, the radial stress decreases with increasing the nonhomogenous index. The converse behavior is observed for radial coordinate higher than this value. The previous mentioned behavior can be observed in Figure 4 at r=0.37.

Disk with free stress edges

By substitution of displacement expression obtained from Equation (17) into (19), we can apply the following stress boundary conditions:

$$\sigma_r = \frac{E(r)}{1-v^2}\left[\frac{du}{dr} + v\frac{u}{r} - (1+v)\alpha T\right]$$

$$\begin{cases} \sigma_r(r=a) = 0 \\ \sigma_r(r=b) = 0 \end{cases} \qquad (19)$$

This boundary condition simulates free rotation of a FG disk. Figure 5 shows the radial displacement of a disk with free stress edges for different values of non-homogenous index (*n*). As shown in Figure 5, the radial displacement decreases with increasing of the nonhomogenous index. This behavior is because of increasing the plate rigidity with increasing the non-homogenous index. Figures 6 and 7 show the radial and circumferential stresses for a disk with two free stress edges.

DISCUSSION

Here the important achieved results are stated and classified as follows:

1. For the first time, true thermoelastic analysis of a FG rotating disk is obtained in the present paper. Achieved differential equation shows that the obtained differential equation that was presented in literature (Hosseini et al., 2007) is incorrect. For this reason, results that extract from that paper are unreliable. In this paper, true thermoelastic formulation of FG rotating disk is investigated by using ADM.
2. In the first kind of boundary condition (fixed edges), it can be understood that the radial displacement at the entire of thickness is positive (outward). In this kind of boundary condition, the radial and circumferential stresses at the inner half of thickness are positive and conversely at the outer half of thickness are negative.

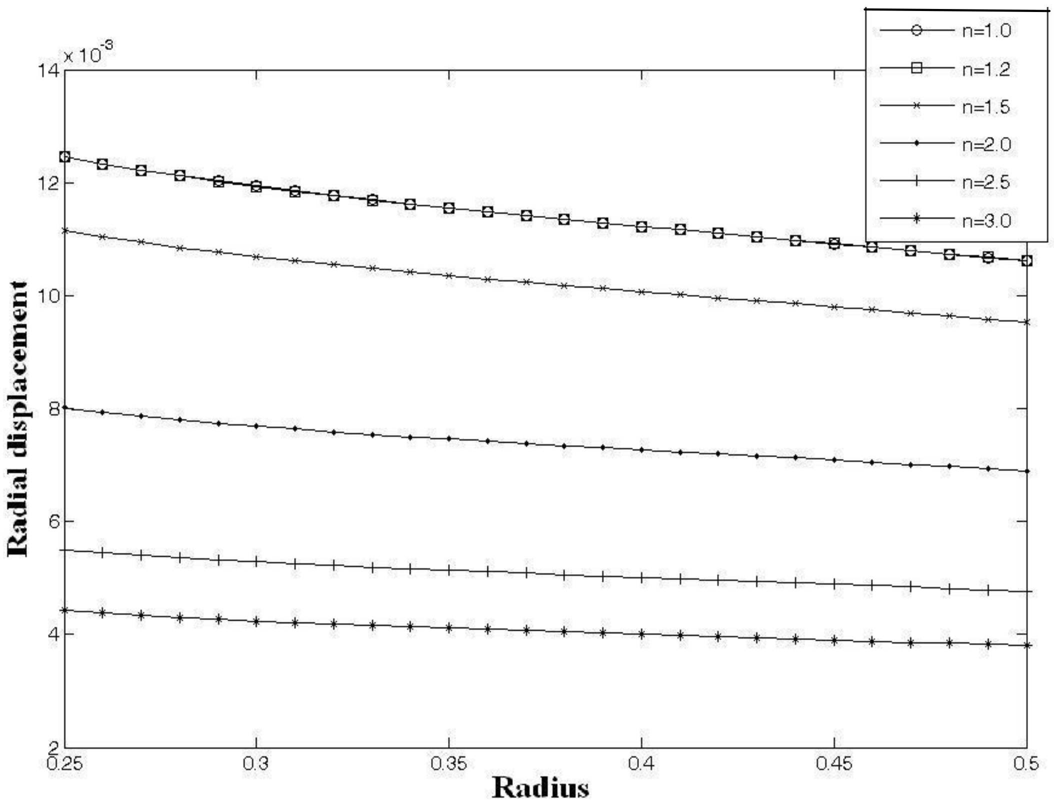

Figure 5. The radial displacement of a disk with free stress edges at inner and outer radii for diiferent value of n.

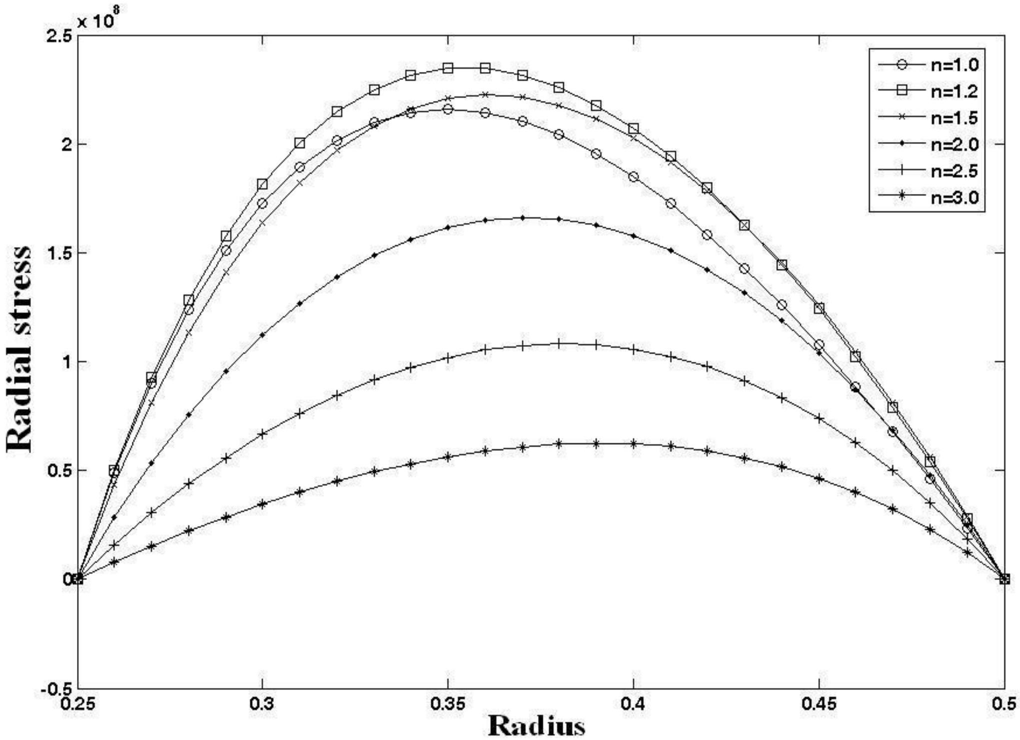

Figure 6. The radial stress of a disk with free stress edges at inner and outer radii for different value of n.

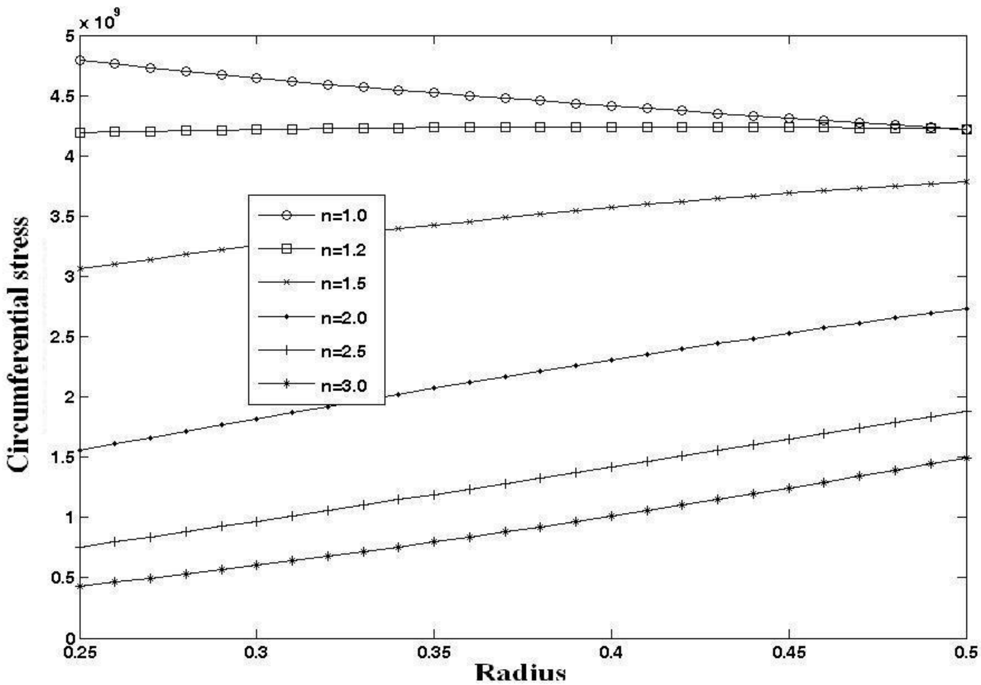

Figure 7. The circumferential stress of a disk with free stress edges at the inner and outer radii for diiferent value of n.

3. In the second kind of boundary condition (free stress edges), it can be understood that with increasing the value of non-homogenous index *n*, the radial displacement decreases. This condition approximately is valid for the radial and circumferential stresses. Except for the first two value of n, for other values of n, radial and circumferential stresses decreases with increasing of the non-homogenous index n.

4. As shown in figures related to first type of boundary conditions, maximum displacement of disk is located at the middle of disk thickness, approximately. Furthermore, this result shows that by changing of the nonhomogenous index of material, maximum value of radial displacement decreases.

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APPENDIX 1

Details of procedure of Equation (6)

$$\frac{d\sigma_r}{dr}+\frac{\sigma_r-\sigma_\theta}{r}+\rho(r)r\omega^2=0 \qquad (1)$$

$$r\frac{dE}{dr}\frac{1}{1-v^2}[\frac{du}{dr}+v\times\frac{u}{r}-(1+v)\alpha T]+\frac{rE}{1-v^2}[\frac{d^2u}{dr^2}+v\{\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}\}-(1+v)\frac{d(\alpha T)}{dr}]$$
$$+\frac{E}{1-v^2}[\frac{du}{dr}+v\frac{u}{r}-(1+v)\alpha T]-\frac{E}{1-v^2}[v\frac{du}{dr}+\frac{u}{r}-(1+v)\alpha T]+\rho r^2\omega^2=0 \qquad (2)$$

$$r\frac{dE}{dr}[\frac{du}{dr}+v\times\frac{u}{r}-(1+v)\alpha T]+rE[\frac{d^2u}{dr^2}+v\{\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}\}-(1+v)\frac{d(\alpha T)}{dr}]$$
$$+E(1-v)[\frac{du}{dr}-\frac{u}{r}]+(1-v^2)\rho r^2\omega^2=0\rightarrow$$
$$rE\frac{d^2u}{dr^2}+[r\frac{dE}{dr}+Ev+E(1-v)]\frac{du}{dr}+[v\frac{dE}{dr}-\frac{vE}{r}-\frac{E(1-v)}{r}]u$$
$$-r\frac{dE}{dr}(1+v)\alpha T-rE(1+v)\frac{d(\alpha T)}{dr}+(1-v^2)\rho r^2\omega^2=0\rightarrow$$
$$rE\frac{d^2u}{dr^2}+[r\frac{dE}{dr}+E]\frac{du}{dr}+[v\frac{dE}{dr}-\frac{E}{r}]u+rE[\frac{du}{dr}\frac{d^2u}{dr^2}+v\frac{1}{r^2}u\frac{du}{dr}]-r\frac{dE}{dr}(1+v)\alpha T$$
$$-rE(1+v)\frac{d(\alpha T)}{dr}+(1-v^2)\rho r^2\omega^2=0 \qquad (3)$$

$$\frac{d^2u}{dr^2}+[\frac{1}{E}\frac{dE}{dr}+\frac{1}{r}]\frac{du}{dr}+[\frac{v}{rE}\frac{dE}{dr}-\frac{1}{r^2}]u-\frac{1}{E}\frac{dE}{dr}(1+v)\alpha T-(1+v)\frac{d(\alpha T)}{dr}+\frac{(1-v^2)\rho r\omega^2}{E}=0 \qquad (4)$$