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Full Length Research Paper

Pre-service mathematics teachers' conceptions about the relationship between continuity and differentiability of a function

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Our purpose in this paper is to investigate and show how pre-service mathematics teachers think about the continuity and differentiability of functions given in the form of both graphical and symbolic representations and how they link these two basic and important calculus concepts. The study was conducted in two phases. First, the participants were asked to complete several tasks involving determination of whether a function is continuous or not, at given points and the differentiability of the function at those points. Second, in light of the collected data, clinical interviews were done with several students in order to better understand their thinking and reasoning. Analysis of both quantitative and qualitative data revealed some junctions in the findings of this study. From these two sources of data, we were able to construct a picture of the students' conceptual link between continuity and differentiability of a function. The results confirmed that students demonstrated difficulties determining the continuity and differentiability of a functions at given points and making connections between limit, continuity and differentiability both in symbolic and graphical representations.

Key words: Continuity, differentiability, limit, function, misconception, misunderstanding.

INTRODUCTION

The subject matter preparation of teachers of mathematics has been a widespread concern. There is evidence to show that teachers with less developed subject matter knowledge tend to focus primarily on mathematical procedures rather than the idea underlying them (Leinhardt and Smith, 1984; Hill, Rowan and Ball, 2005). Therefore, our focus in this present study is to gain a better understanding on how pre-service mathematics teachers relate two important concepts in calculus: continuity of a function and its differentiability. Review of the existing literature showed that over the past two decades researchers have focused on the ways that students think about functions (Dubinsky and Harel, 1992; Leinhardt et al., 1990; Romberg et al., 1993). Although students' knowledge about limit (Sierpinska, 1987; Bezuidenhout, 2001; Brown, 2004;), continuity (Vinner, 1987; Cornu, 1991; Vinner, 1992; Lauten, Graham and Ferrini-Mundy, 1994; Wilson, 1994; Bezuidenhout, 2001), differentiability and derivation (Orton, 1983; Monk and Nemirovsky, 1994; Aspinwalll et al., 1997; Asiala, Cottrill, Dubinsky and Schwingendorf, 1997; Bezuidenhout, 1998; Baker, Cooley, and Trigueros, 2000; Bezuidenhout, 2001) have been studied extensively in most of the previous research studies. It is not quite clear how students represent the relationship between continuity of a function to its differentiability.

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Believing that there is still need fro gaining new information, this study builds on existing studies (Tall and Vinner 1981; Artigue, 1991; Viveros and Sacristan, 2002) in the literature by incorporating the notion of exploring, describing, and analyzing the pre-service mathematics teachers' ways of making connections between these two concepts.

Moreover, another purpose of this study is to identify pre-service mathematics teachers' common misconceptions and inadequate understandings by examining how they mentally organize the information about the concepts. Numerous researchers have discussed and documented student difficulties and misunderstandings. In 1992, at the seventh International Congress on Mathematical Education in Québec, a working group focused on students' difficulties in calculus. Problems in the teaching and learning of calculus have also been discussed in a series of publications by the Mathematical Association of America. Tall and Schwarzenberger (1978), Tall and Vinner (1981), Davis and Vinner (1986), and Cornu (1991) have described many of these problems. We believe that present investigation enhances the existing literature on students' misconceptions and inadequate understandings on calculus from this perspective.

Present study also investigates the problems mentioned above through analysis of students' use of both symbolic and graphical representations of functions. Specially, one of our purposes is to obtain more information about students' use of two different representational registers, as symbolic and graphical, in order to determine the continuity of a function, the differentiability of a function and the relationship between them.

BACKGROUND

Although there is plenty of literature pertaining to students' understanding of various aspects of functions, existing literature related to students' understanding of the relationship between continuity and differentiability of a function is limited. Review of the literature is designed to provide examples of previous research studies related to students' understanding of the relationship between continuity, values, limit, and differentiability of a function from different perspectives. First, research studies discussing the link between students' conceptions of functions, and their conceptions of continuity are provided. Most of the research on this matter agreed upon that students had difficulties in combining ideas of continuity and functions. Second research studies revealing the relationship between ideas of continuity and the functional representations are provided in order to get more insights about the sources of students' misconceptions about continuity in different representational registers. Third, existing research demonstrating the connections between students' conceptions of limit, and their conceptions of continuity are presented. We believe that there is still much to be learned about the ways that conceptions of limit tie into conceptions of continuity. Finally, yet importantly, literature discussing the relationship between student thinking about continuity and their thinking about differentiability is presented. Since mathematically these two ideas are linked, it is important to understand how students think about the connections. Therefore, we included examples of existing research focusing on relationship between students' ideas of continuity and different functional representations.

Continuity and function values

There are many studies demonstrating the connections between students' conceptions of functions, and their conceptions of continuity. Some research studies have consistently shown students' difficulties in separating out the differences between considerations of continuity and function values. For instance, Bezuidenhout (2001) reported that students in his study did not consider the continuity as a function property rather they focused on the value of function at a given point. Vinner (1992), in his study, also came up with similar results. He conducted a study with 406 college calculus students. In the study, students were given functions, which were represented either graphical or symbolic, and asked to determine whether given functions were continuous. Most of the students' responses demonstrated a common belief that "for a function to be continuous is the same as being defined and to be discontinuous is the same as being undefined at a certain point" (p. 205). Moreover, Lauten et al. (1994) also conducted a case study with a calculus student to understand the student's ideas of continuity and functions.

Student's work on function construction tasks indicated that students had difficulties in combining ideas of continuity and functions. They reported that the student seemed to have difficulties in imagining a discontinuous function on a connected domain. Wilson (1994) reported similar results in his work with a pre-service teacher who claimed that discontinuous graph shown to her was wrong and a function can be graphed so that it is continuous.

Continuity in different representational registers

Some researchers have investigated the relationship between ideas of continuity and the functional representtation. According to Ferrini-Mundy and Graham (1994), for example, when the problems are presented with different functional representations, students use different ways of thinking. Lauten et al. (1994) stated that students handled the same problems in different ways when problems were presented in different representations such as graphical and algebraic. Students in their study showed disconnects between the symbolic and graphical representations of function, when researchers asked them to describe their ideas about continuity. Authors stated that type of functional representation seemed to have an influence on students' interpretations while dealing with the continuity of function.

For example, Cornu (1991) expressed that meaning of the word "continuity" in everyday language evoked misconceptions on students' interpretations of continuity when the function represented graphically. According to the author students have tendency to equate the connectedness of a graph with the continuity of a function. Researchers then stated that teachers often use the phrases "being in one piece" or "drawn without taking the pencil off the paper" in order to give insight into the concept of continuity while working on the graph of a function. In so doing, they cause confusions on the mathematical ideas of continuity and connectedness. Tall and Vinner (1981) have also found similar results in their study. Their results demonstrated that students' concept images of continuity were affected by the informal language that is used to describe continuity such as a graph having no 'gaps', being 'all in one piece'. When determining the continuity, many students related the concept of continuity to the connectedness of the graph such as having breaks, jumps or gaps.

Continuity and limit

There are some research studies demonstrating students' tendency to equate the existence of a limit with continuity. One of these demonstrations appears in the study of Williams (1991). A questionnaire in this study was administered to 341 second-semester calculus students at a university. Then in-depth interviews were done with ten (10) of the participants in order to better understand the students' limit conceptions. Results of this study revealed that students had difficulties in connecting the continuity and the existence of limits. For example, some students thought that limit was appropriate only for discontinuous functions. Thus, taking a limit of a continuous function was not possible. Consequently, for these students, continuity and existence of limits were inequitable rather than being related.

Continuity and differentiability

According to Tall and Vinner (1981), students had trouble distinguishing between the concepts of differentiability and continuity. For example, one student who claimed that a function was discontinuous since there was a sudden change in slope. Meel (1998) conducted a research with 26 calculus students and the results of the study revealed that most of the students had beliefs that continuity was necessary and it is sufficient condition for deciding differentiability of a function. Tall and Vinner (1981) in their study obtained opposite results revealing that student used non-differentiability as a reason of discontinuity of a function. Both of the mentioned studies was agreed upon which students had trouble determining the relationship between these two ideas. Bezuidenhout (2001) also reported similar findings that students were unable to distinguish between continuity and differenttiability. Baker et al. (2000) conducted a study with 41 college students, who had completed two semesters of calculus. The students were asked to sketch a graph of a continuous function satisfying several criteria related to the first and second derivatives on given intervals. It was observed that the students were unable to conceive what a function is, which was continuous but not differentiable, might look like.

The authors then reported that the students were familiar with functions that were both continuous and differentiable. Although some students who were able to construct such a function, according to authors, they were not able to understand how removing the continuity might affect the graph of the function they have drawn. As a result, they did not fully understand how differentiability and continuity are related.

METHOD

Participants

This study was conducted with 59 pre-service mathematics teachers who were enrolled in General Mathematics course, which is one of the fundamental undergraduate courses offered in a mid-size public university in Turkey. Course content basically consists of a review of algebraic operations; equations and inequalities; functions and functional notations; graphs; inverse functions; linear, quadratic and rational functions; absolute value; radicals; exponential and logarithmic functions; systems of equations and inequalities; limit and continuity; and differentiation.

Procedure

Qualitative and quantitative data were collected concurrently during the study for the purpose of triangulating the data for better confirmation and corroboration of the findings, even though primary method in this study was analysis of qualitative data collected through in dept clinical interviews with the selected participants. Creswell (2003) stated that, any convergence of findings from both qualitative and quantitative data results in strengthening the researcher's claims. There were some junctions of findings found from the analysis of two data sets in this Study. Chart 1 represents the data collection strategy that was used in this study expressed by Creswell (2003) as Concurrent triangulation strategy.

Research instruments

In order to obtain more detailed information on pre-service

Chart 1. Concurrent triangulation strategy (Adopted from Creswell (2003) p. 214).

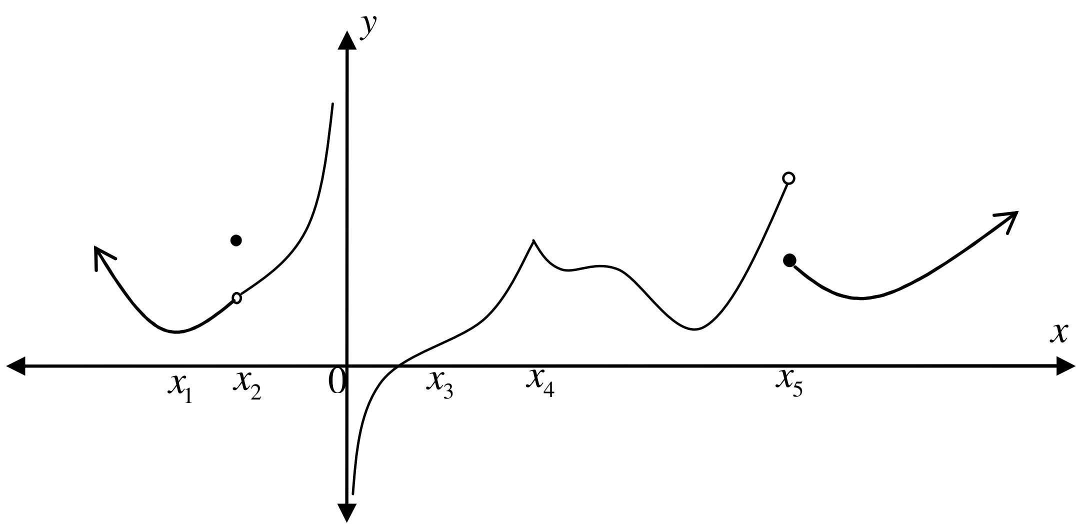


Figure 1. Graphical task.

mathematics teachers' understandings and misconceptions', if there is any, about the relationship between limit, continuity and differentiability qualitative and quantitative data were collected through the questionnaire which consisted of four open ended questions. For establishing the validity of questionnaire, a number of questions were given to a group of five experts. These experts determined whether the questions in the questionnaire were appropriate for the purpose of obtaining information on pre-service mathematics teachers' understandings and misconceptions' on the relationship between limit continuity and differentiability. All of these experts were from the area of mathematics education. Questions with at least 80% agreement were used in this questionnaire. Reliability of this questionnaire was established by administering the questionnaire in a pilot study that was held in one semester before the data collection. In this pilot study questions were tested in terms of consistency in students' responses given to each item and questions were also determined whether they were understandable or not. Some of the questions used in study are given in the following

a. At what numbers is discontinuous? Why?
b. At what numbers is not differentiable? Why?

In Figure 1, the graph of f function is given.

Function f is given as, $f(x) = \begin{cases} 3x & x \leq 1 \\ x^2 - 1 & 1 < x \leq 3 \\ 8 & 3 < x \end{cases}$

a. Investigate whether function f is continuous or not at points $x = 1$ and $x = 3$.

b. Investigate whether function f is differentiable or not at points $x = 1$ and $x = 3$.

FINDINGS

In this section, students' responses to the specific questions were analyzed from different viewpoints in terms of their understanding of the relationship between continuity and differentiability. More specifically, students' solution strategies and their approaches when working on functions presented both graphically and symbolically were investigated in order to better understand their misunderstandings and misconceptions in different representational systems.

Students' understandings of continuity

Students in this study were given several function questions in which functions are presented graphically and symbolically in order to acquire clear ideas about their understanding of continuity. It was observed that more than half of the students (63% or 37 out of 59) failed to determine all of the discontinuous points on a graphically presented function. Only 22 students determined all of the points ($x_2, 0\, and\ x_5$) correctly. Twenty-eight students gave partly correct responses. Twenty-two out of these 28 students, who gave partly correct responses, determined two points (x_2, x_5), and three students determined the point "0" only, one student said the function is discontinuous at points "$0, x_5$",but one student also said only point "x_5" and another student said only point "x_2".

The remaining nine students were either unable to determine correct points where the function is discontinuous or they did not respond at all. When the students' explanations were analyzed, we noted that most of the students who correctly determined all of the points have also given acceptable reasons. For example, one student stated, "there is a removable discontinuity at point x_2. $\lim_{x \to x_2^-} f(x) = \lim_{x \to x_2^+} f(x) \neq f(x_2)$ At the point $x = 0$, function *f* has no limit, since $\lim_{x \to 0^-} f(x) = +\infty\ ve\ \lim_{x \to 0^+} f(x) = -\infty$ therefore it is not continuous. At the point x_5 , $\lim_{x \to x_5} f(x)$ does not exist so it is discontinuous." As seen in the excerpt, student also talked about types of the discontinuities such as removable or not and implied the definition of continuity. Another student stated, "The function is discontinuous at points $x_2\ x_5\ and\ 0$, since when the graph is drawn, we should lift the pencil of the paper." It was clearly seen that students refereed the graphical definition of continuity of a function, which is accepted as source of misconception by some researchers. This definition used 12% of all students.

Analysis of the students' responses and written explanations revealed that students who gave partly correct responses by determining either two or one discontinuous points (28 out of 59) and students who could not determine the correct points at all (9 out of 59) had misunderstandings and lack of knowledge on the concepts of limit and continuity.

It was revealed that seven students thought that if the limit exist at a point $x = a$, it is sufficient for the function f to be continuous at that point. Followings are some excerpt from student written responses:

"The function is discontinuous at points x_2, x_5, since right and left limits are not equal at these points."
"The function is discontinuous at points x_2, x_5, since the function is not defined and right and left limits are not equal at these points."
"In order to determine whether a function is continuous or not at some points, we have to look at limits from right and left. When we approach to the points x_2, x_5 from right and left, we obtain different values. Therefore we can say the function is discontinuous at points x_2, x_5."
"The function is discontinuous at points x_2, x_5, since function has two different values at these points."

There are 13 students who thought that being defined at a point is sufficient for a function to be continuous at that point. Followings are some examples of students' written responses:

"The function is discontinuous at the points where the function is undefined. Therefore, this function is discontinuous at only point $x = 0$."
"The function is discontinuous at the point $x = 0$, since graph doesn't go through this point."
"The function is discontinuous at the point x_2, since function is not defined at that point."
"The function is discontinuous at points x_2, x_5, since function f doesn't have values at these points."

These excerpts taken from the students expressions indicated that the students have some difficulties determining the correct points where the function is discontinuous. Because of the lack of prior knowledge on continuity, it was revealed that these students could not apply all of the conditions (for a function $f\ (A \subset R,\ f : A \to R)$ to be continuous at the point $x_0 \in R$ (a) function f should be defined at the point x_0, (b) $\lim_{x \to x_0} f(x)$ should be exist and (c) there must exist a $\lim_{x \to x_0} f(x) = f(x_0)$ equation) for a function to be continuous at a point. For example, one student's expression such as "The function is discontinuous at points x_2, x_5 since the function is undefined and limits from right and left are not equal" indicated student's tendency

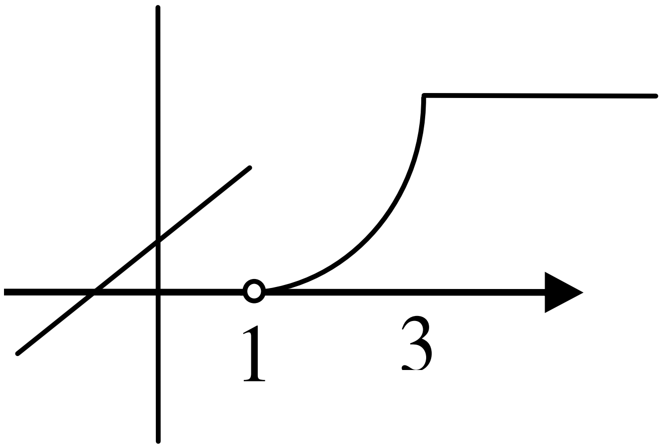


Figure 2. Continuous function at point $x=3$.

to relate being defined to being continuous and to relate being discontinuous to being undefined at a certain point as well (Figure 1). In addition, when student's response were analyzed, we found out that student has some difficulties in finding the domain of function. Excerpts taken from the student's response revealed that this student thought that for a function to be defined at a point $x=x_0$; corresponding value for this point should be on the graph. Another student stated, "Points where the function is undefined are discontinuous. Therefore, the function is discontinuous at only point $x=0$." From this expression, we can say that this student has misunderstanding by relating the continuity to being defined at a point.

It was observed that students in this study were more successful in determining the discontinuous points ($x=1$ and $x=3$) in a partial function given symbolically than determining the discontinuous points in a function given graphically. Most of the students (44 out of 59) used algebraic methods and definition of continuity in order to determine the discontinuous points while some students (9 out of 59) drew the graph of the function to determine those points. There were six students who did not respond at all. Analysis of the students' responses revealed that 40 (68%) students correctly determined all of the points where the function is discontinuous. Of these 40 students, five of them determined the correct points by drawing the graph of the function. Following is an example showing one student's solution strategy and explanation by graphing the function:

"Considering the definition of continuity, we should be able to draw the whole graph without holding the pencil. Therefore, at the point $x=1$, when approached from right and left it can be seen that f is discontinuous. $\lim_{x\to 3^-} f(x)=8,\ \lim_{x\to 3^+} f(x)=8$ Therefore at the point $x=3$ the function is continuous" (Figure 2).

Same student gave a correct answer to the question seen in Figure 1 by stating, "The function is discontinuous at point $x=0$, since it is undefined. The function is also discontinuous at points $x_2,\ x_5$." Likewise, other four students who correctly determined the discontinuous points by graphing gave correct answers to the question where the function is given graphically. Moreover, when expressions of students, who gave correct responses by using the definition of continuity, analyzed we can see that explanations generally referred to the formal definition of continuity. For example, one student stated "at point $x=1$, $\lim_{x\to 1^-} f(x)=3,$ $\lim_{x\to 1^+} f(x)=0,\quad f(1)=3$ and $\lim_{x\to 1^-} f(x)\neq \lim_{x\to 1^+} f(x)\neq f(1)$, therefore the function is discontinuous at point $x=1$. $\lim_{x\to 3^-} f(x)=8,\ \lim_{x\to 3^+} f(x)=8,\ \ f(x)=8$ And $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)=f(3)$ therefore the function is continuous at point $x=3$." From this statement, we can say that this student knows the definition of continuity of a function and can apply this definition to the problem presented. However, it is very interesting that the same student stated,

"The function is discontinuous at points $x_2,\ x_5$ since function f has no values at these points (undefined)", while working on the function given graphically (Figure 1). One of the reasons for this conflict may stem from students' tendency to use symbolic representations rather than graphical representations. Another reason may come from students' lack of conceptual understanding of the definition of continuity.

Students' understandings of the relationship between continuity and differentiability

According to the findings, only 5 out of 59 students participated in this study gave correct responses by stating that the function cannot be differentiable at points $x_2,\ 0,\ x_5\ and\ x_4$. In addition, two students determined the points $x_2,\ \ x_5\ and\ x_4$; one student said the function could not be differentiable at points $x_2,\ \ x_4$.

It was noted that these students considered the cusp points besides the points where the function is discontinuous. Five students stated that the function could not be differentiable at a point x_4, which is a cusp point on the graph, and two students did not give any response at all.

There were 35 students stating that the function cannot be differentiable at the points where it is discontinuous. Nine of the remaining students expressed that the function cannot be differentiable at the points where it has no limit or undefined.

Considering the analysis of both students' responses and interviews about their understanding of differentiability, it can be said that students have some misconceptions and misunderstandings. Following is the explanation of a student who gave a correct response:

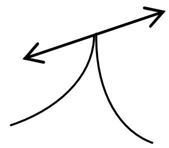
"The function cannot be differentiable at points $x_2, 0, x_4, x_5$ since the function is discontinuous at points x_2, 0, x_5 and we can draw infinite number of tangent lines at point x_4. Only one tangent line can be drawn at a point where the function is differentiable."

Following is an explanation comes from another student who also gave correct response:

"Every differentiable function is continuous. However, not every continuous function can be differentiable. For the function to be differentiable, it should be continuous. Therefore, the function cannot be differentiable at points x_2, 0, x_5. Moreover, the function cannot be differentiable at point x_4 since the slope cannot be found."

From these expressions, it can be clearly said that these students understood and applied the fundamental theorem of calculus which states "if f is differentiable at a, then f is continuous at a". However, considering the analysis of both expressions of first student and interview of second student it can be seen that students have some misconceptions and misunderstandings about cusp point and were affected by their prior geometry knowledge. Following is a part of interview conducted with second student:

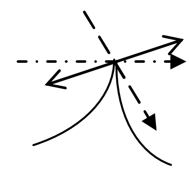
R: In your response, you said that function cannot be differentiable at point x_4 since the slope cannot be found. What did you mean by slope here?
S: When we say derivative it means slope. For example, when we say derivative at a point we look at the slope of the tangent line drawn through that point.
R: Cannot a tangent line be drawn at point x_4 ?
S: It cannot be drawn through point x_4 since there is a peek point like here…ummm …for example



a tangent line can be drawn like this [drawing a tangent line].

R: the tangent line you drew has a slope.

S: We can also draw



tangent lines like these [drawing couple tangent lines].
R: What does this mean?
S: I mean there must be only one value. We can draw however, we like now. Therefore, there is no slope.
R: Isn't there any slope?
S: We can find infinite number of slopes. I mean, infinite numbers of tangent lines can be drawn at point x_4. These infinite numbers of tangent lines have infinite numbers of slopes. Therefore, we cannot determine which of these slopes corresponds to the value of derivative at that point. Some students who stated that there was no derivative at the point x_4 were affected by the notion of nonexistence of "0". These students stated that since the slope at point x_4 was "0" there is no derivative at that point. Following is an excerpt taken from an interview with one of the participants.
R: In your response to the problem, you stated that the function can not be differentiable at point x_4. Could you explain why?
S: The function cannot be differentiable at point x_4 since the slope at this point is zero.
R: If the slope is zero, does this mean there is no derivative at that point?
S: Yes. If the slope is zero, there is no derivative. Zero means nonexistence.

More than half of the students (59%) misunderstood one of the fundamental theorem in calculus "If f is differentiable at a, then f is continuous at a." Most of them thought that the reverse of this theorem "If f is continuous at a, then f is differentiable at a." is also true.
These students have also stated that the function cannot be differentiable at only points where it is discontinuous. This misconception was observed in both the work of students who correctly determined all the discontinuous points and work of other students. Followings are some examples taken from students explanations:

"There is no derivative at points $x_2, 0, x_5$. Function must be continuous at these points."
"The function f should be continuous at the points $x_2, 0, x_5$ in order it to be differentiable."
"There is no derivative at discontinuous points $x_2, 0, x_5$ "
"The function cannot be differentiable at points x_2, x_5 since it is not continuous at these points."
"The function cannot be differentiable at point x_2 since it is discontinuous at this point."

Data obtained from interviews also indicated the same misconception among students. Following is an example excerpt taken from an interview with one of the participant of the study:

R: When you look at the graph, could you say that the function is differentiable at the point x_4 ?
S: The limit at the point x_4 is known. Ummm… I am really confused now. As a matter of fact that the function is continuous at this point and it can be differentiable. And the function is also defined at this point.
R: Could you explain the kind of relationship between continuity and differentiability?
S: I could not remember the relationship now. Ummm… The function can be differentiable if it is continuous. But not all of the differentiable functions are continuous. I am a little confused. Ummm…Yes. Yes. The function can be differentiable if it is continuous.

Nine students were thought that if a function was defined and had a limit at a point it would be sufficient for that function to have derivative at that point. Here are some examples taken from students' explanations:

"…there is no derivative at points $0, x_5$. Right and left limits must be equal."
"…function f cannot be differentiable at points 0 and x_5 since limits get different values when approached from right and left."
"…function cannot be differentiable at point $x = 0$. The function should be defined at that point to be differentiable. Function is not defined at point $x = 0$. Therefore it has no derivative at this point."

Data obtained from the interviews demonstrated similar expressions. Excerpt is an example taken from interview with one of the participants which is as follows:

R: in the graphical representation of function, you stated that there is no derivative at points $0, x_5$ since there is no limit. Could you explain why so is?
S: Why didn't I say point x_2? Since right and left limits exist at point x_2. It can be discontinuous but this does not mean that it has no limit at that point. There is limit here [x_2]…for a function to be differentiable at a point, there must be limit at that point. There is no limit at point $x = 0$. ummm…at the point x_5…from right and left …I think limits are different so there is no limit we can say.
R: Did you mean that having a limit in a point x_0 is sufficient for a function to be differentiable at that point?
S: I can say yes.

Considering the students' responses regarding differentiability at points $x = 1$ and $x = 3$, it was noted that students gave similar responses in partial function represented symbolically. Six of the students stated that at points $x = 1$ and $x = 3$, there is no derivative. Three of these students used symbolic definition of derivative; one of them used numerical definition of derivative and two of them found by drawing graph of the function. Following are some examples of students' responses:
"*There is no derivative at point* $x = 1$ *since* $f'(1^-) = 3, f'(1^+) = 2x = 2, 3 \neq 2$ *and there is no derivative at point* $x = 3$ *since* $f'(3^-) = 2x = 6, f'(3^+) = 0, 6 \neq 0$ *.*"
"The function f is not differentiable at point $x = 1$ since it is not continuous at that point but it is continuous at point $x = 3$.

$$f'(3^-) = \lim_{x \to 3^-} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3^-} \frac{x^2 - 1 - 8}{x - 3}$$

$$\lim_{x \to 3^-} \frac{(x-3).(x+3)}{x-3} = 6$$

$$f'(3^+) = \lim_{x \to 3^+} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3^+} \frac{x^2 - 1 - 8}{x - 3}$$

$$\lim_{x \to 3^-} \frac{8 - 8}{x - 3} = 0$$

And $f'(3^-) \neq f'(3^+)$, although the function f is continuous at point 3, it is not differentiable."

Seven of the remaining students did not respond to the question and 46 of them gave inaccurate responses. Considering the responses of students who gave wrong answers and interviewed with, it was noted that there were misconceptions, which were seen in the graphical representation of function, seen in the function given by symbolically also. Thirty-four out of 46 students who gave incorrect responses indicated that the function f is not differentiable since it is not continuous at point $x=1$ and the function is differentiable at point $x=3$ since it is continuous at that point. Similarly, six students stated that the function is not differentiable at point $x=1$ because it has no limit at that point and it is differentiable at point $x=3$ since it has limit at that point. As we can see in students responses, these students thought that having limit for a function is sufficient to be differentiable. Following are some expressions of students who thought that existence of limit and continuity at the point are satisfactory for a function to be differentiable:

"The function is not continuous at point $x=1$ therefore it cannot be differentiable at that point and the function is continuous at point $x=3$ then it can be differentiable at that point."

"…at point $x=1$, $\lim_{x\to 1^+} f(x)=0$, $\lim_{x\to 1^-} f(x)=3$ so the function cannot be differentiable at that point. But at point $x=3$, $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^{+-}} f(x)=8$ thus it can be differentiable."

"…at point $x=1$, the function is discontinuous for this reason it cannot be differentiable. However, at point $x=3$, the function is continuous which means limits from right and left exist and they are equal to the value of the function at that point. In that case, the function can be differentiable."

"While, at point $x=1$, f is not continuous so cannot be differentiable, at point $x=3$ f is continuous and defined so f can be differentiable."

DISCUSSION

Information assessed in this research was intended to be foundational for building and connecting the major ideas of calculus such as continuity, limit and differentiability. Therefore, we believe the findings of this research suggest a need to examine the development of students' understanding of concepts of limit, continuity and differentiability and the students' emerging misconceptions while setting up the links between these concepts during their study of calculus.

According to findings, pre-service mathematics teachers have different ideas and some difficulties in determining the discontinuous points on the graphical representation of a given function. Prior research studies revealed (Ferrini-Mundy and Graham, 1994) that the type of functional representation has influence on students' interpretations while dealing with the continuity of a function and students have different approaches depending on the type of representation of a function. It was clearly observed that most of the participants in this study preferred algebraic approach, which involves checking all of the necessary conditions for continuity in order to determine discontinuous points in a function given graphically. In addition, they were also more successful in finding discontinuous points in symbolic representation of the function comparing to determining discontinuous points in graphical representation. This finding is in agreement with the findings of previously conducted studies (Asiala et al., 1997; Dreyfus and Eisenberg, 1982; Eisenberg, 1991, 1992) showing students' strong tendencies to use and prefer algebraic and analytical methods while working on functions. Accordingly, we believe that domination of procedural knowledge and procedural methods for solving the problems in middle and high school mathematics curriculums make students more skill masters by using algebraic and symbolic ways to solve problems. On the other hand, findings of this resent study was revealed that some of the pre-service mathematics teachers consistently prefer to draw graph of the symbolically given function in order to determine discontinuous points. These students have also used the informal definition of continuity "A continuous function as a function whose graph has no hole or break" while working on the function either given graphically or symbolically and whose graphs were drawn by students . It seems that students generally find this informal definition very easy to use and apply on a graph of a function to determine the continuity at points. However, previous research studies (Tall and Vinner, 1981; Vinner, 1992; Ferrini-Mundy and Graham, 1991) indicated that the use of this informal definition of continuity causes some misconceptions.

One of the important findings of this study is that pre-service mathematics teachers have some misconceptions in determining the continuity. Approximately 25% of the participants thought that existence of limit in a point or even being defined in the point are sufficient for a function to be continuous at that point. In this matter, this study have similar findings with prior research studies (Bezuidenhout, 2001; Lauten et al., 1994; Vinner, 1992) consistently showing students' difficulties in separating out the differences between considerations of continuity and function values. Besides, some of the participants thought that in order for a function to be defined at a point $x=x_0$, corresponding value of that point should be on the graph of the function. In other words, these students thought that if the value of a point exists but is not on the graph of a function, the function is undefined at that point.

Another important observation was that most of the pre-service mathematics teachers in this study have a common misconception on the relationship between continuity and differentiability of a function in a point by thinking that the reverse of the theorem “every differentiable function in a point is also continuous at that point” is also true. In other words, most of the participant in this study thought that if a function is continuous at a point then it is differentiable at that point. For instance while working on the graphically presented function, some participants stated that the function is differentiable at point x_4 since the function is continuous at that point and this is sufficient for it to be differentiable. Similarly, while working on the function presented symbolically, some participants used same reasoning in order to determine differentiability of the function at points $x=1$ and $x=3$ by first looking for the continuity at these points. They thought that the continuity at a point is sufficient for a function to be differentiable at that point. A well-known result confirmed the results of previously conducted studies (Viveros and Sacristan, 2002; Bezuidenhout, 2001; Meel, 1998; Tall and Vinner, 1981). As observed in our study, prior to the studies mentioned above, have also revealed similar results stating that some students’ consistently thought that continuity and differentiability were equivalent.

Findings of this research also revealed that pre-service mathematics teachers have misconceptions about differentiability of a given function at a cusp. It was observed that participants investigated the differentiability of function at cusp point in both graphically and symbol-lically given functions by drawing the graph of function. Although most of the students stated that the given function could not be differentiable at the cusp, it was observed that they mostly used inappropriate reasoning. One of the reasoning they used was that infinite numbers of tangent lines, which could be drawn at cusp, point and there were infinite numbers of slopes. They stated that they could not determine which one corresponds to the derivative at that point. Here we can say that students did not understand the idea of derivative conceptually and they established their reasoning on prior knowledge of geometry concepts. This finding is in agreement with the findings of Vinner’s (1991) study, investigating the first-year college calculus students’ conceptions of tangent. He reported that students seem to be strongly affected by the concept image of tangent they have learned in geometry. Other reasoning participants used was that no tangent lines could be drawn at cusp point therefore slope could not be found. Another reasoning used was that slope of the tangent line could be “0” and this means nonexistence of slope, therefore, there is no derivative. Here we can say that students were affected by the meaning of “0” they learned while studying on natural numbers.

In light of findings of this research and existing literature, it is clearly observed that students have some difficulties understanding important concepts of calculus such as limit, continuity and differentiability and the relationship between these concepts. Further studies about identifying the underlined reasons of students’ misconceptions and misunderstandings on calculus concepts are needed.

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