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Full Length Research Paper

Contractions on Hilbert space with the smallest local unitary spectra

Cesim Temel

Department of Mathematics, Faculty of sciences, Yuzuncu Yil University, 65080 Van, Turkey.
E-mail: cesimtemel@yahoo.com. Tel: +90 0432 225 10 25-29.

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Let T be a contraction on a complex Hilbert space H, let $\sigma_T(x)$ be the local spectrum of T at $x \in H$, and let $\sigma_T(x) \cap \Gamma$ be the local unitary spectrum of T at x; $\Gamma = \{z \in C : |z| = 1\}$. We show that if $\sigma_T(x) \cap \Gamma$ is of Lebesgue measure zero, then $\lim_{n\to\infty} \|T^n x\| = \|x_T^u\|$, where x_T^u is the unitary part of x in the canonical decomposition of H with respect to T.

Key words: Hilbert space, contraction, local spectrum.

INTRODUCTION

Let H and K be complex Hilbert space and let $B(H,K)$ be the space of all bounded linear operators from H to K; for $H = K$, we simply write $B(H)$. An operator $T \in B(H)$ is a *contraction* if $\|T\| \leq 1$. If $T \in B(H)$ is a contraction, then for every $x \in H$, $\lim_{n\to\infty} \|T^n x\|$ exists and is equal to $\inf_{n\geq 0} \|T^n x\|$. A contraction $T \in B(H)$ is said to be a C_1-contraction if $\inf_{n\geq 0} \|T^n x\| > 0$, for every $x \in H \setminus \{0\}$. For an arbitrary $T \in B(H)$, we denote as usual by $\sigma(T)$ the spectrum of T and by $R(z,T) = (zI - T)^{-1}$ the resolvent of T. In this paper, we will $D = \{z \in C : |z| < 1\}$, $\Gamma = \{z \in C : |z| = 1\}$, and $A(D)$ denotes for the disc-algebra. If $T \in B(H)$ is a contraction, then the spectrum of T lies in $\overline{D}$. The set $\sigma_u(T) := \sigma(T) \cap \Gamma$ is called the unitary spectrum of T.

For an arbitrary $T \in B(H)$ and any $x \in H$, we define $\rho_T(x)$ to be the set of all $\lambda \in C$ for which there exists a neighborhood U_λ of λ with $u(z)$ analytic on U_λ having values in H, such that $(zI - T)u(z) = x$ on U_λ. This set is open and contains the resolvent set $\rho(T)$ of T. By definition, the local spectrum of T at x, denoted by $\sigma_T(x)$ is the complement of $\rho_T(x)$, so it is a closed subset of $\sigma(T)$. If $T \in B(H)$ is a contraction and $x \in H$, then the set $\sigma_T(x) \cap \Gamma$ will be called the *local unitary spectrum* of T at x. Consider the case where U is a unitary operator on H. Let $E(\cdot)$ be the spectral measure of U. For given $x \in H$, let μ_x be the vector-measure defined on the Borel subsets of Γ by $\mu_x(\Delta) = E(\Delta)x$. One can easily see that $\sigma_U(x) = supp(\mu_x)$.

Generally, the local spectrum of an operator $T \in B(H)$ may be very "small" with respect to its usual spectrum. Indeed, let σ be a "small" part of $\sigma(T)$ such that both σ and $\sigma(T) \setminus \sigma$ are closed sets. Let P_σ be the spectral projection associated with σ and let $H_\sigma = P_\sigma H$. We know that H_σ is a (closed) T-invariant subspace and $\sigma(T|_{H_\sigma}) = \sigma$. Now, we can readily verify that $\sigma_T(x) \subset \sigma$ for every $x \in H_\sigma$.

Let $T \in B(H)$ be a contraction and let $x \in H$. We can see that $\xi \in \rho_T(x) \cap \Gamma$ if and only if $R(z,T)x$ $(|z|>1)$ admits an analytic extension to some neighborhood of ξ. It follows that if $\xi \in \rho_T(x) \cap \Gamma$ for every $x \in H$, then $\xi \in \rho(T)$. Hence, we have

$$\sigma_u(T) = \bigcup_{x \in H} (\sigma_T(x) \cap \Gamma).$$

Note that there exist a contraction $T \in B(H)$ and $x \in H$ such that $\sigma_T(x) \cap \Gamma = \varnothing$, but $\sigma_u(T) = \Gamma$. Indeed, let $H^2(K)$ be the Hardy space of K-valued analytic functions on D and let S be the unilateral shift operator on $H^2(K)$; $(S_K f)(\lambda) = \lambda f(\lambda)$. Its adjoint, the backward shift, is given by:

$$(S^* f)(\lambda) = \frac{f(\lambda) - f(0)}{\lambda}, \quad f \in H^2(K).$$

It is easy to verify that for every $f \in H^2(K)$ and $z \in C$ with $|z| > 1$,

$$(zI - S^*)^{-1} f(\lambda) = \frac{z^{-1} f(z^{-1}) - \lambda f(\lambda)}{1 - \lambda z}.$$

Hence $\sigma_{S_K^*}(f) \cap \Gamma$ consists of all $\xi \in \Gamma$ such that f has no analytic extension to a neighborhood of ξ. It follows that if f admits an analytic extension across the unit circle, then $\sigma_{S_K^*}(f) \cap \Gamma = \varnothing$. However, $\sigma_u(S_K^*) = \Gamma$. Note also that for every nonzero $f \in H^2(K)$, $\sigma_{S_K}(f) = \overline{D}$.

Recall that a contraction $T \in B(H)$ is said to be completely non-unitary if it has no proper reducing subspace on which it acts as a unitary operator. As is well known (Nikolski, 1986), if $T \in B(H)$ is a contraction, then there exists a canonical decomposition (with respect to T) of the space H into two T-invariant subspaces: $H = K \oplus L$ such that: i) K and L reduce T; ii) $S := T|_K$ is a completely non-unitary contraction; iii) Z $U := T|_L$ is a unitary operator, where the subspace L is defined by:

$$L = \{x \in H : \|T^n x\| = \|T^{*n} x\| = \|x\|\} \quad n \in \aleph.$$

The operator S (respectively U) will be called completely non-unitary (unitary) part of T. According to this decomposition, every $x \in H$ can be written as $x = x_T^c + x_T^u$. The vector x_T^c (respectively x_T^u) will be called *completely non-unitary (unitary) part of* x.

It can be seen that if $T \in B(H)$, $\lim_{n\to\infty} \|T^n\| = 0$ if and only if $\sigma_u(T) = \varnothing$. Generally, the asymptotic behavior of the sequence $\{T^n\}_{n \in \aleph}$ is frequently related to unitary spectrum of the underlying operator. This is well illustrated by the following classical result of Nagy-Foias (Nagy and Foias, 1966). If the unitary spectrum of a completely non-unitary contraction $T \in B(H)$ has Lebesgue measure zero, then $\lim_{n\to\infty} \|T^n x\| = 0$ for all $x \in H$ (the proof based on unitary dilation arguments). In this paper, we address the problem whether local and quantitative versions of the Nagy-Foias Theorem hold. For related results see (Allan and Ransford, 1989; Batty et al., 1998; Mustafayev, 2010).

RESULTS

The following theorem is the main result of this paper.

Theorem 1

Let $T \in B(H)$ be a contraction and let $x \in H$ be such that $\sigma_T(x) \cap \Gamma$ is of Lebesgue measure zero. Then, we have:

$$\lim_{n\to\infty} \|T^n x\| = \|x_T^u\|,$$

where x_T^u is the unitary part of x in the canonical decomposition of the space H with respect to T.

For the proof, we need the following lemmas.

Lemma 1

Let $T \in B(H)$ be a contraction, let E be a T-invariant subspace, and let $\pi : H \to H/E$ be the canonical mapping. Then, the following assertions hold:

a) $\sigma_{T|_E}(x) \cap \Gamma = \sigma_T(x) \cap \Gamma$ for every $x \in E$;

b) $\sigma_T(x_T^c)\cap\Gamma\subset\sigma_T(x)\cap\Gamma$, where x_T^c is the completely non-unitary part of $x\in H$ in the canonical decomposition of H.

c) $\sigma_{\hat{T}}(\pi x)\subset\sigma_T(x)$ for every $x\in H$, where $\hat{T}$ is the induced mapping; $\hat{T}\circ\pi=\pi\circ T$.

Proof

a) Let $x\in E$. It is easy to see that $\sigma_T(x)\subset\sigma_{T|_E}(x)$, and so

$$\sigma_T(x)\cap\Gamma\subset\sigma_{T|_E}(x)\cap\Gamma.$$

For the reverse inclusion, let an arbitrary $\xi\in\rho_T(x)\cap\Gamma$ be given. Then, there exists a neighborhood U_ξ of ξ with $u(z)$ analytic on U_ξ having values in H, such that $(zI-T)u(z)=x$ on U_ξ. Since

$$u(z)=R(z,T)x=\sum_{n=0}^{\infty}z^{-n-1}T^nx\in E,$$

for all $z\in U_\xi$ with $|z|>1$, we have $\pi u(z)=0$ for all $z\in U_\xi$ with $|z|>1$. By uniqueness theorem, $\pi u(z)=0$ for all $z\in U_\xi$, so that $u(z)\in E$. Thus, we have $(zI-T|_E)u(z)=x$ on U_ξ. This shows that $\xi\in\rho_{T|_E}(x)\cap\Gamma$.

b) Let $H=K\oplus L$ be the canonical decomposition of H and let $S=T|_K$. It follows from a) that

$$\sigma_T(x_T^c)\cap\Gamma=\sigma_S(x_T^c)\cap\Gamma.$$

It remains to show that $\sigma_S(x_T^c)\subset\sigma_T(x)$. If $\lambda\in\rho_T(x)$, then there exists a neighborhood U_λ of λ with $u(z)$ analytic on U_λ, having values in H, such that $(zI-T)u(z)=x$ on U_λ. Let P be the orthogonal projection from H onto K. Then, we have $(zP-PT)u(z)=x_T^c$. Since $PT=TP=SP$, we obtain $(zI-S)Pu(z)=x_T^c$. This shows that $\lambda\in\rho_S(x_T^c)$.

c) If $\lambda\in\rho_T(x)$, then there exists a neighborhood U_λ of λ with $u(z)$ analytic on U_λ having values in H, such that $(zI-T)u(z)=x$ on U_λ. It follows that $(z\pi-\pi\circ T)u(z)=\pi x$ on U_λ. Consequently, we have $(zI-\hat{T})\pi u(z)=\pi x$ on U_λ. This shows that $\lambda\in\rho_{\hat{T}}(\pi x)$.

Recall that $V\in B(H)$ is called an *isometry* if $\|Vx\|=\|x\|$ for all $x\in H$. It is well known that if V is a non-unitary isometry, then $\sigma(V)=\overline{D}$. Recall also that $x\in H$ is a *cyclic vector* of $T\in B(H)$, if the set $\{T^nx:n=0,1,2,...\}$ spans the whole space H.

Lemma 2

If $V\in B(H)$ is an isometry and $x\in H$ is a cyclic vector of V, then

$$\sigma_u(V)=\sigma_V(x)\cap\Gamma.$$

Proof

Assume that $VH=H$, that is, V is a unitary operator. We must show that $\sigma(V)=\sigma_V(x)$. By Spectral Theorem, there exists a positive measure μ on Γ such that the operator M on $L^2(\Gamma,\mu)$ defined by $Mf=e^{it}f$ is unitary equivalent to V. Let χ_Δ denotes the characteristic function of any Borel subset Δ of Γ and let 1 be the constant one function on Γ. Then, we have $\sigma(V)=\mathbf{supp}(\mu)$ and $\sigma_V(x)=\mathbf{supp}(v)$, where v is a vector measure on Γ that is defined by $v(\Delta)=\chi_\Delta 1$. Since $\|v(\Delta)\|=\mu(\Delta)$, we have $\mathbf{supp}(\mu)=\mathbf{supp}(v)$ and so, $\sigma(V)=\sigma_V(x)$.

Assume that $VH\neq H$. In this case $\sigma(V)=\overline{D}$, so that $\sigma_u(V)=\Gamma$. It is enough to show that $\sigma_V(x)=\overline{D}$. Let $K=H\ominus VH$. By Wold's Decomposition Theorem (Nagy and Foias, 1966), there exists a decomposition $H=H_0\oplus H_1$ such that H_0 and H_1 reduce V, $V_0=V|_{H_0}$ is unitary and $V_1=V|_{H_1}$ is unitary equivalent to the unilateral shift operator S_K on $H^2(K)$. Let $x=x_0+x_1$, where $x_0\in H_0$ and $x_1\in H_1$. Since x_1 is a cyclic vector of V_1, $x_1\neq 0$, so that $\sigma_{V_1}(x_1)=\overline{D}$. It remains to show that $\sigma_{V_1}(x_1)\subset\sigma_V(x)$. If $\xi\in\rho_V(x)$,

then there exists a neighborhood U_ξ of ξ with $u(z)$ analytic on U_ξ having values in H, such that $(zI - V)u(z) = x$
on U_ξ. Let P_1 be the orthogonal projection from H onto H_1. Then, we have $(zP_1 - P_1V)u(z) = x_1$. Since $P_1V = VP_1 = V_1P_1$, we obtain $(zI - V_1)P_1u(z) = x_1$. This shows that $\xi \in \rho_{V_1}(x_1)$.

Lemma 3

Let $T \in B(H)$ *be a* C_1*-contraction and let* $x \in H$*, if* $f \in A(D)$ *vanishes on* $\sigma_T(x) \cap \Gamma$*, then* $f(T)x = 0$.

Proof

By Nagy-Foias Theorem (Nagy and Foias, 1966), there exist an isometry V and a quasi-affinity X on H intertwining T and V; $XT = VX$. First, we claim that

$$\sigma_V(Xx) \subset \sigma_T(x). \qquad (1)$$

If $\lambda \in \rho_T(x)$, then there exists a neighborhood U_λ of λ with $u(z)$ analytic on U_λ having values in H, such that $(zI - T)u(z) = x$ on U_λ. It follows that

$$(zX - XT)u(z) = Xx \quad (z \in U_\lambda)$$

Consequently, we have $(zI - V)Xu(z) = Xx$ on U_λ. This shows that $\lambda \in \rho_V(Xx)$

Set

$$K = \overline{span}\{V^n Xx : n = 0,1,2,...\},$$

and

$$L = \overline{span}\{T^n x : n = 0,1,2,...\}.$$

Since $V^n Xx = XT^n x$ $(n \in \aleph)$, the operator $X|_L$ is a quasi-affinity from L to K and

$$(V|_K)X|_L = (X|_L)T|_L. \qquad (2)$$

Also, since Xx is a cyclic vector of $V|_K$, by Lemma 2

$$\sigma_u(V|_K) = \sigma_{V|_K}(Xx) \cap \Gamma.$$

On the other hand, taking into account Lemma 1 a) and (1), we can write

$$\sigma_{V|_K}(Xx) \cap \Gamma = \sigma_V(Xx) \cap \Gamma \subset \sigma_T(x) \cap \Gamma.$$

Hence, we have

$$\sigma_u(V|_K) \subset \sigma_T(x) \cap \Gamma. \qquad (3)$$

We see that under the hypotheses of the Lemma, the Lebesgue measure of $\sigma_T(x) \cap \Gamma$ is necessarily zero. It follows from (3) that $\sigma_u(V|_K)$ has Lebesgue measure zero and therefore, $V|_K$ is a unitary operator. Since $f \in A(D)$ vanishes on $\sigma_T(x) \cap \Gamma$, it follows that f vanishes on $\sigma(V|_K)$, and so $f(V)K = \{0\}$. Using now the identity (2), we can write $Xf(T)L = \{0\}$. In particular, we have $Xf(T)x = 0$. Since X has zero kernel, we obtain that $f(T)x = 0$.

Lemma 4

Let $T \in B(H)$ be a C_1-contraction and let $x \in H$. *If* $\sigma_T(x) \cap \Gamma$ *is of Lebesgue measure zero, then* $\|T^n x\| = \|T^{*n} x\| = \|x\|$ for all $n \in \aleph$.

Proof

Set $M = \sigma_T(x) \cap \Gamma$. Let us define a mapping $h : C(M) \to H$ as a following way: Take a function $f \in C(M)$. By Rudin-Carleson Theorem (Beauzamy, 1988), there exists a function $\overline{f} \in A(D)$ such that $\overline{f}(\xi) = f(\xi)$ for all $\xi \in M$, and

$$\|\overline{f}\|_{A(D)} = \sup_{\xi \in M} |f(\xi)|. \qquad (4)$$

Set $h(f) = \overline{f}(T)x$. By Lemma 3, h is a well-defined linear mapping. On the other hand, it follows from von Neumann inequality and the identity (4), the mapping h is bounded. Note also that if $f, g \in C(M)$, then $h(fg) = \overline{f}(T)\overline{g}(T)x$. Assume that the functions f_{-1}, f_0

and f_1 on M is defined by $f_{-1}(\xi)=\xi^{-1}$, $f_0(\xi)=1$ and $f_1(\xi)=\xi$. Then, we have

$$x=h(f_0)=h(f_{-1}f_1)=\overline{f}_{-1}(T)\overline{f}_1(T)x.$$

Set $S=\overline{f}_{-1}(T)$. Then, S is a contraction on H which commutes with T. Since $\overline{f}_1(T)=T$, we have $STx=x$ so that $ST^nx=T^{n-1}x$ for all $n\in ℵ$. It follows that

$$\|T^{n-1}x\|=\|ST^nx\|\le\|T^nx\|\le\|T^{n-1}x\|.$$

Thus, $\|T^nx\|=\|x\|$ for all $n\in ℵ$. We know (Nagy and Foias, 1966) that if T is an arbitrary contraction and z is an eigenvector of T for the eigenvalue $\lambda=1$, then z is also an eigenvector of T^* for the eigenvalue $\lambda=1$. Since ST is a contraction and $STx=x$, we have $S^*T^*x=x$. It follows that $\|T^{*n}x\|=\|x\|$ for all $n\in ℵ$.

We are now able to prove the Theorem 1

Proof of Theorem 1

Let $H=K\oplus L$ be the canonical decomposition of H and let $S=T|_K$ be the completely non-unitary part of T. Let $x=x_T^c+x_T^u$, where x_T^c is the completely non-unitary and x_T^u is the unitary part of x. Let us show that $\lim_{n\to\infty}\|T^nx_T^c\|=0$. For this reason, set

$$K_0=\left\{x\in K:\lim_{n\to\infty}\|S^nx\|=0\right\}.$$

Let $\pi:K\to K/K_0$ be the canonical mapping and let $\hat{S}:K/K_0\to K/K_0$ the induced mapping; $\hat{S}\circ\pi=\pi\circ S$. First, we claim that $\hat{S}$ is a C_1-contraction. For this, it is enough to show that for every $x\in K$,

$$\lim_{n\to\infty}\|\hat{S}^n\pi x\|=\lim_{n\to\infty}\|S^nx\|.$$

Indeed, let

$$\alpha=\lim_{n\to\infty}\|\hat{S}^n\pi x\|=\lim_{n\to\infty}\|S^nx+K_0\|.$$

Then, we have $\alpha\le\lim_{n\to\infty}\|S^nx\|$. On the other hand, for an arbitrary $\varepsilon>0$ there exist $k\in ℵ$ and $y\in K_0$, such that $\|S^kx-y\|\le\alpha+\varepsilon$, which implies $\|S^{n+k}x-S^ny\|\le\alpha+\varepsilon$, for all $n\in ℵ$. It follows that

$$\|S^{n+k}x\|\le\|S^{n+k}x-S^ny\|+\|S^ny\|\le\alpha+\varepsilon+\|S^ny\|.$$

As $n\to\infty$, we obtain $\lim_{n\to\infty}\|S^nx\|\le\alpha+\varepsilon$, so that $\lim_{n\to\infty}\|S^nx\|\le\alpha$. Further, it follows from the identity $\hat{S}^*=S^*|_{K_0^\perp}$ that $\hat{S}$ is a completely non-unitary contraction. Using Lemma 1 c), a), and b), respectively; we have

$$\sigma_{\hat{S}}(\pi x_T^c)\cap\Gamma\subset\sigma_S(x_T^c)\cap\Gamma=\sigma_T(x_T^c)\cap\Gamma\subset\sigma_T(x)\cap\Gamma.$$

It follows that $\sigma_{\hat{S}}(\pi x_T^c)\cap\Gamma$ has Lebesgue measure zero. Since $\hat{S}$ is a completely non-unitary C_1-contraction, by Lemma 4, $\pi x_T^c=0$, and so

$$\lim_{n\to\infty}\|T^nx_T^c\|=\lim_{n\to\infty}\|S^nx_T^c\|=0.$$

Also, since $\|T^nx_T^u\|=\|x_T^u\|$ for all $n\in ℵ$, we have that

$$\lim_{n\to\infty}\|T^nx\|=\lim_{n\to\infty}\|T^nx_T^c+T^nx_T^u\|=\lim_{n\to\infty}\|T^nx_T^u\|=\|x_T^u\|.$$

CONCLUSION

It is easy to verify that if $T\in B(H)$, then $\lim_{n\to\infty}\|T^n\|=0$ if and only if $\sigma_u(T)=\varnothing$. In general, the asymptotic behavior of the sequence $\{T^n\}_{n\in ℵ}$ is frequently related to unitary spectrum of the underlying operator. This is well illustrated by the classical result of Nagy-Foias (Nagy and Foias, 1966). If the unitary spectrum of a completely non-unitary contraction $T\in B(H)$ has Lebesgue measure zero, then $\lim_{n\to\infty}\|T^nx\|=0$ for all $x\in H$. In this note we show that if $\sigma_T(x)\cap\Gamma$ is of Lebesgue measure zero, then $\lim_{n\to\infty}\|T^nx\|=\|x_T^u\|$. Consequently, local and quantitative version of the well known Nagy-Foias Theorem is proved.

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