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Haar Wavelet-Picard technique for fractional order nonlinear initial and boundary value problems

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In this article, a technique called Haar wavelet-Picard technique is proposed to get the numerical solutions of nonlinear differential equations of fractional order. Picard iteration is used to linearize the nonlinear fractional order differential equations and then Haar wavelet method is applied to linearized fractional ordinary differential equations. In each iteration of Picard iteration, solution is updated by the Haar wavelet method. The results are compared with the exact solution.

Key words: Fractional differential equations, Wavelet analysis, Caputo derivative, Haar wavelets, Picard iteration.

MSC 2010: 34Bxx, 65Lxx.

INTRODUCTION

Haar wavelet is the lowest member of the Daubechies family of wavelets and is convenient for computer implementations due to the availability of explicit expression for the Haar scaling and wavelet functions (Daubechies, 1990). Operational approach is pioneered by Chen (Chen and Hsiao, 1997) for uniform grids. The basic idea of Haar wavelet technique is to convert differential equations into a system of algebraic equations of finite variables.

Boundary value problems are considerably more difficult to deal with than the initial value problems. The Haar wavelet method for boundary value problems is more complicated than for initial value problems. Second-order boundary value problems are solved in Siraj-ul-Islam et al. (2010) by the Haar wavelets; they considered the six sets of different boundary conditions for the solution. Boundary value problems for fractional differential equations are solved in Mujeeb and Khan (2012) which considers the numerical solution by the Haar wavelet for different boundary value problems of fractional order.

The Picard approach (Bellman and Kalaba, 1965) is used to linearize the individual or system of nonlinear ordinary and partial differential equations. In this paper, we consider the case of fractional order nonlinear ordinary differential equations which contain various forms of nonlinearity. The main aim of the present paper is to get the numerical solutions of nonlinear fractional order initial ad boundary value problems over a uniform grids with a simple method based on the Haar wavelets and Picard technique.

THE HAAR WAVELETS

The Haar functions contain just one wavelet during some

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subinterval of time and remains zero elsewhere and are orthogonal. The Haar wavelets are useful for the treatment of solution of differential equations (Chen and Hsiao, 1997). The ith Haar wavelet $h_i(x), x \in [a,b]$ is defined as

$$h_i(x) = \begin{cases} 1, & a+(b-a)\frac{k}{m} \leq x < a+(b-a)\frac{k+0.5}{m}, \\ -1, & a+(b-a)\frac{k+0.5}{m} \leq x < a+(b-a)\frac{k+1}{m}, \\ 0, & otherwise. \end{cases} \quad (1)$$

where $i = 2^j + k + 1, j = 0,1,2,\cdots,J$ is the dilation parameter, where $m = 2^j$ and $k = 0,1,2,\cdots,2^j - 1$ is translation parameter. J is maximal level of resolution and the maximal value of i is $2M$, where $M = 2^J$. In particular $h_1(x) = \chi_{[a,b]}(x)$, where $\chi_{[a,b]}$ is the characteristic function on interval [a, b], is the Haar scaling function. For the Haar wavelet, the wavelet collocation method is applied. The collocation points are usually taken as $x_j = \left(\frac{b-a}{2}\right)\left(\frac{j+0.5}{2M}\right) + \left(\frac{b+a}{2}\right), j = 1,2,\cdots,2M.$

Fractional integral of the uniform Haar wavelets

Any function $y \in L_2[a,b]$ can be represented in term of the uniform Haar series

$$y(x) = \sum_{i=1}^{\infty} b_i h_i(x), \quad (2)$$

where b_i are the Haar wavelet coefficients given as $b_i = \int_{-\infty}^{\infty} y(x) h_i(x) dx.$

The Riemann-Liouville fractional integral of the Haar scaling function is given as

$$I_a^{\alpha} h_1(x) = \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}, \quad (3)$$

and

$$I_a^{\alpha} h_i(x) = \frac{1}{\Gamma(\alpha+1)} \begin{cases} (x-a(i))^{\alpha}, & a(i) \leq x < b(i); \\ (x-a(i))^{\alpha} - 2(x-b(i))^{\alpha}, & b(i) \leq x < c(i); \\ (x-a(i))^{\alpha} - 2(x-b(i))^{\alpha} + (x-c(i))^{\alpha}, & x \geq c(i) \end{cases} \quad (4)$$

CONVERGENCE ANALYSIS

As our work is based on Picard technique and Haar wavelet method, we thus analyze the convergence of both schemes.

Convergence of Picard technique

Consider the nonlinear second order differential equation

$$y''(x) = f(y), y(0) = y(b) = 0. \quad (5)$$

Application of Picard technique to (5) yields

$$y''_{n+1}(x) = f(y_n), \quad y_{n+1}(0) = y_{n+1}(b) = 0. \quad (6)$$

Let $y_0(x)$ be some initial approximation. Each function $y_{n+1}(x)$ is a solution of the linear equation (6), where y_n is always considered known and is obtained from the previous iteration.

According to Picard iteration:

$$|y_{n+1} - y_n| \leq k|y_n - y_{n-1}|, \text{ where } k < 1. \quad (7)$$

This shows that there is linear convergence, if there is convergence at all.

Convergence of Haar wavelet method

Let $y(x)$ be a differentiable function and assume that $y(x)$ have bounded first derivative on $(0,1)$, that is, there exists $K > 0$; for all $x \in (0,1)$

$$|y'(x)| \leq K.$$

Haar wavelet approximation for the function $y(x)$ is given by $y_M(x) = \sum_{i=1}^{2M} b_i h_i(x).$

Babolian and Shahsavaran (2009) gave L_2-error norm for Haar wavelet approximation, which is

$$\|y(x) - y_M(x)\|^2 \leq \frac{K^2}{3}\frac{1}{(2M)^2},$$

or

$$\|y(x) - y_M(x)\| \leq O\left(\frac{1}{M}\right), \quad (8)$$

where $M = 2^J$ and J is the maximal level of resolution. From inequality (8), we conclude that error is inversely proportional to the level of resolution. Equation (8) ensures the convergence of Haar wavelet approximation at higher level of resolution, that is, when M is increased.

APPLICATIONS

Here, we solve nonlinear differential equations of fractional order by the Haar wavelets-Picard technique and compare the results with the exact solution. Throughout this work we use Caputo derivatives and for the details of fractional derivatives and integrals we refer the readers to Podlubny (1999).

Example 1: Consider the αth order fractional nonlinear Bratu type equation,

$$ {}^{c}D^{\alpha}y(x) - 2e^{y(x)} = 0, \quad 1 < \alpha \le 2, \tag{9}$$

subject to the initial condition $y(0) = 0, y'(0) = 0.$

The exact solution, when $\alpha = 2$, is given by (Kiymaz, 2010)

$$y(x) = -2\ln(\cos x). \tag{10}$$

Applying the Picard iteration to the Equation 9, we get

$$ {}^{c}D^{\alpha}y_{r+1}(x) = 2e^{y_r(x)}, \ 1 < \alpha \le 2, \tag{11}$$

with the initial condition $y_{r+1}(0) = 0, y'_{r+1}(0) = 0.$

Now we apply the Haar wavelet method to Equation 11; we approximate the higher order derivative term by the Haar wavelet series as

$$ {}^{c}D^{\alpha}y_{r+1}(x) = \sum_{i=1}^{2M} b_i h_i(x). \tag{12}$$

Lower order derivatives are obtained by integrating Equation 12 and using the initial condition

$$y_{r+1}(x) = \sum_{i=1}^{2M} b_i p_{\alpha,i}(x),\ y'_{r+1}(x) = \sum_{i=1}^{2M} b_i p_{\alpha-1,i}(x). \tag{13}$$

Substituting Equations 12 and 13 in Equation 11, we get

$$\sum_{i=1}^{2M} b_i h_i(x) = 2e^{y_r(x)}, \tag{14}$$

with the initial approximation $y_0(x) = 0.$

We fix the order of the differential Equation (9), $\alpha = 2$ and the level of resolution, $J = 5$. The graph in Figure 1 shows the exact and approximate solutions by proposed method at four iterations. The absolute error reduces with the increasing iterations.

Results at fifth iteration of proposed method at fixed level of resolution, $J = 3$ and at different values of α are shown in Figure 2 with exact solution at $\alpha = 2$. Figure 2 showed that numerical solutions converge to the exact solution when α approaches to 2.

Example 2: Consider the fractional nonlinear Duffing equation

$$ {}^{c}D^{\alpha}y(x) + y'(x) + y(x) + y^3(x) = \cos^3(x) - \sin(x), 1 < \alpha \le 2, \tag{15}$$

subject to the initial conditions $y(0) = 1, y'(0) = 0.$

The exact solution, when $\alpha = 2$, is

$$y(x) = \cos(x) \tag{16}$$

Applying the Picard iteration to Equation 15

$$ {}^{c}D^{\alpha}y_{r+1}(x) + y'_{r+1}(x) + y_{r+1}(x) = \cos^3(x) - \sin(x) - y_r^3(x), 1 < \alpha \le 2, \tag{17}$$

with the initial condition $y_{r+1}(0) = 1, y'_{r+1}(0) = 0.$

Applying the Haar wavelet method to Equation (17), we approximate the higher order derivative term by the Haar Wavelet series as

$$ {}^{c}D^{\alpha}y_{r+1}(x) = \sum_{i=1}^{2M} b_i h_i(x) \tag{18}$$

Now to get the Haar wavelet series for lower order derivatives terms we integrate Equation 18 and use the initial condition

$$y_{r+1}(x) = \sum_{i=1}^{2M} b_i p_{\alpha,i}(x) + 1,\ y'_{r+1}(x) = \sum_{i=1}^{2M} b_i p_{\alpha-1,i}(x). \tag{19}$$

Substituting Equations 18 and 19 in Equation 17, we get

$$\sum_{i=1}^{2M} b_i\left[h_i(x) + p_{\alpha-1,i}(x) + p_{\alpha,i}(x)\right] = \cos^3(x) - \sin(x) - y_r(x)^3 - 1, \tag{20}$$

with the initial approximations $y_0(x) = 1, y'_0(x) = 0.$

We fix the level of resolution, $J = 3$ and order of differential Equation (15), $\alpha = 2.$ The exact and numerical solutions by proposed method at different iterations along with the absolute error are shown in Figure 3. We observe that error reduces with increase in iterations.

Exact solution at $\alpha = 2$ and the Haar solution at different values of $\alpha,$ are displayed in Figure 4. It is observed that solutions of fractional nonlinear Duffing Equation (15) converge to the solution of second order nonlinear Duffing equation, when α approaches to 2.

Example 3: Consider the α^{th}order nonlinear Lane-Emden type Equation:

$$ {}^{c}D^{\alpha}y(x) + \frac{2}{x}y'(x) + 8e^{y(x)} + 4e^{\frac{y(x)}{2}} = 0, \ 1 < \alpha \le 2, \tag{21}$$

subject to the initial conditions $y(0) = 0, y'(0) = 0.$

The exact solution, when $\alpha = 2$, is

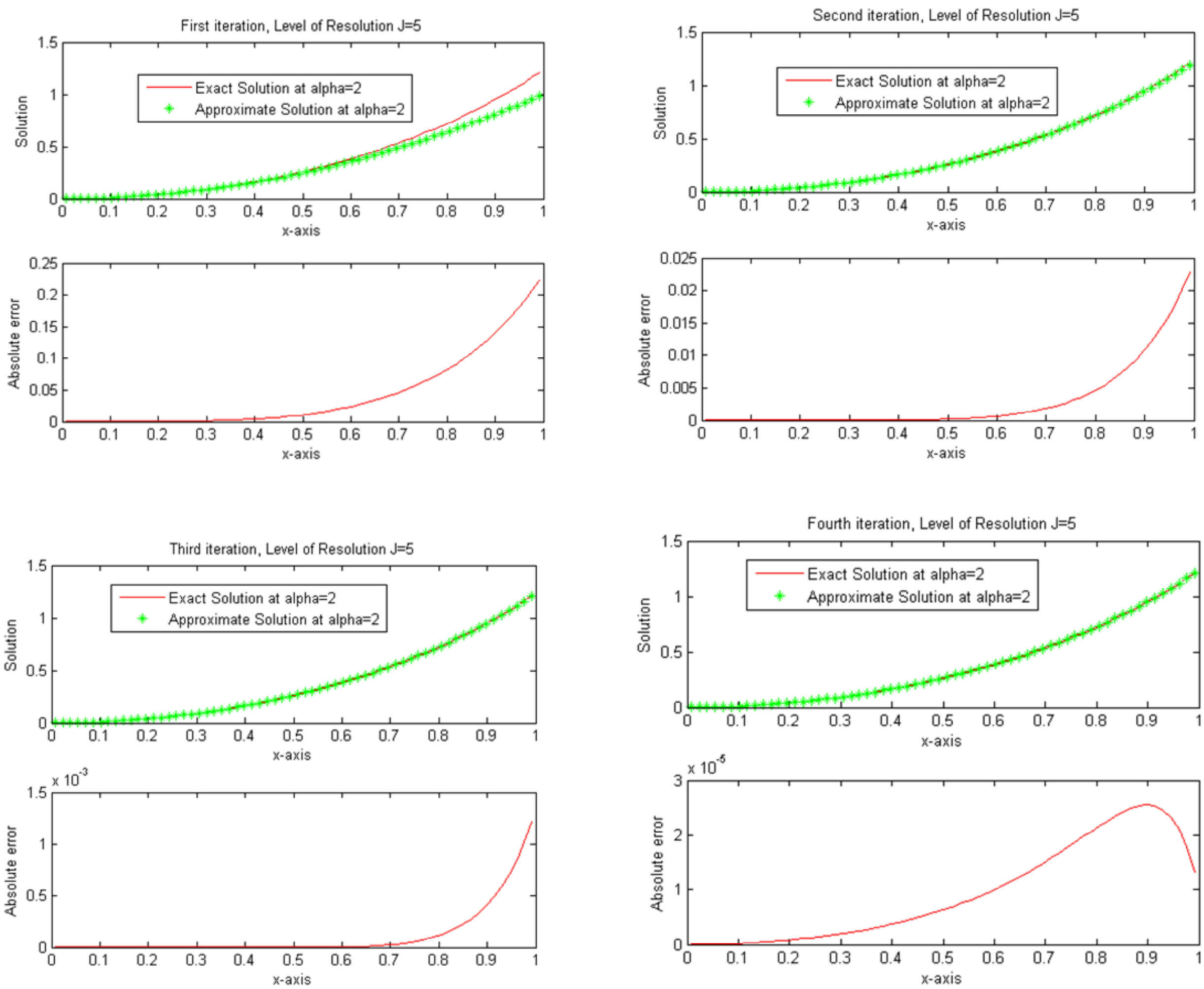

Figure 1. Comparison of exact solution and solutions by Haar wavelet-Picard technique at J = 5; for different iterations, and α=2.

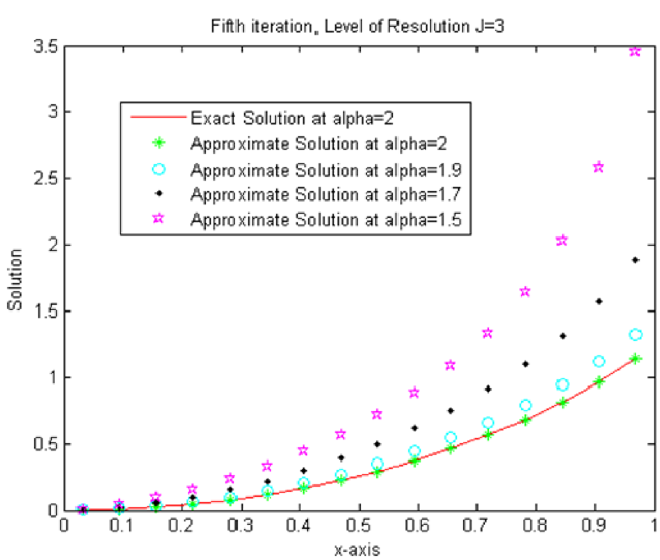

Figure 2. Exact solution at α = 2 and the Haar wavelet-Picard solution at α = 2, α = 1.9, α = 1.7, and α = 1.5.

$$y(x) = -2\ln\left(1+x^2\right). \tag{22}$$

Picard iteration to Equation 21 implies

$${}^cD^\alpha y_{r+1}(x) + \frac{2}{x} y'_{r+1}(x) = -8e^{y_r(x)} - 4e^{\frac{y_r(x)}{2}}, \; 1 < \alpha \le 2, \tag{23}$$

with the initial conditions $y_{r+1}(0) = 0, y'_{r+1}(0) = 0.$

Applying the Haar wavelet method to equation

$${}^cD^\alpha y_{r+1}(x) = \sum_{i=1}^{2M} b_i h_i(x) \tag{24}$$

Lower order derivatives are obtained by integrating Equation 25 and using the initial condition

$$y_{r+1}(x) = \sum_{i=1}^{2M} b_i \, p_{\alpha,i}(x), \tag{25}$$

$$y'_{r+1}(x) = \sum_{i=1}^{2M} b_i p_{\alpha-1,i}(x). \tag{26}$$

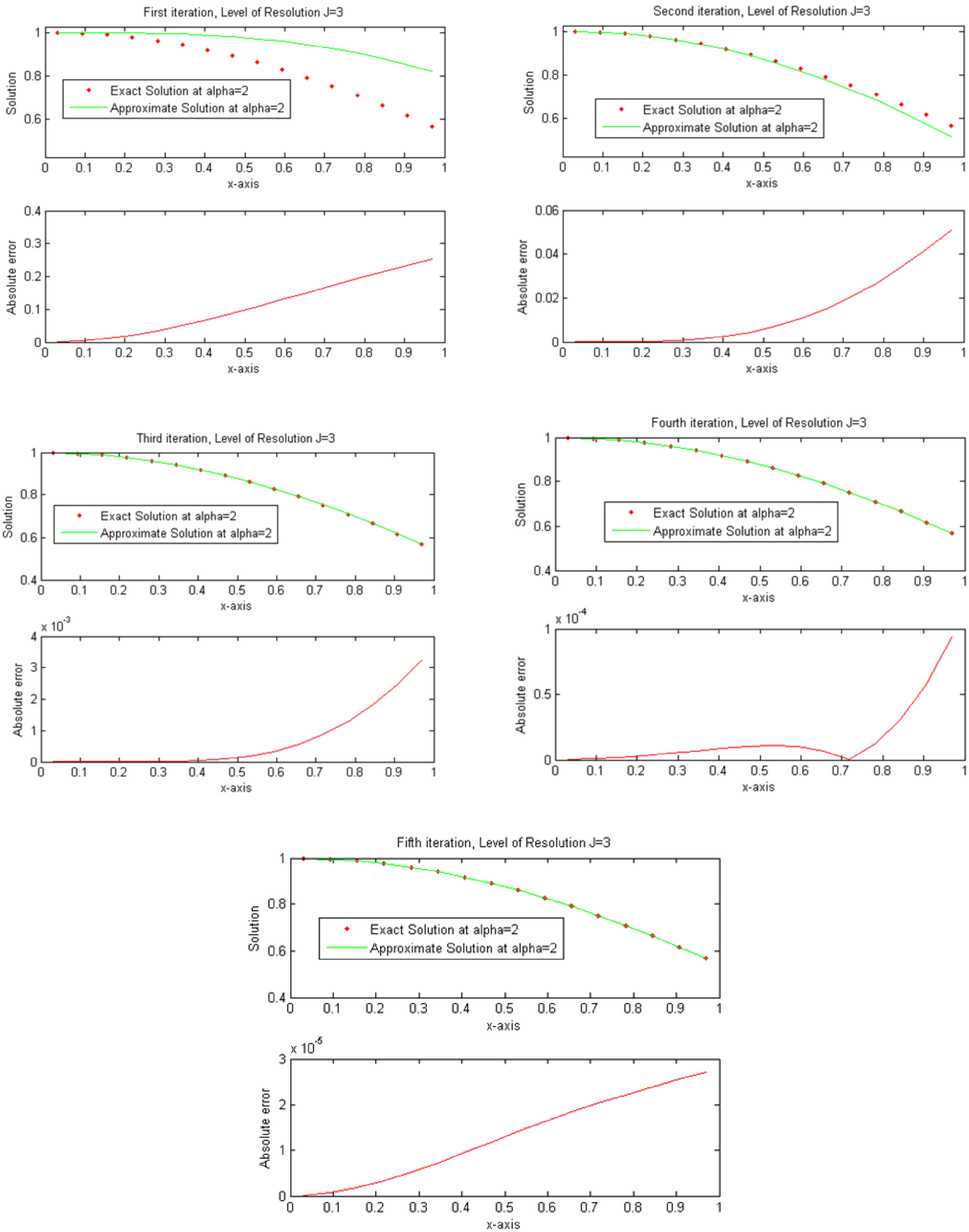

Figure 3. Comparison of exact solution and solutions by Haar wavelet-Picard technique at J = 3, for different iterations, and α =2.

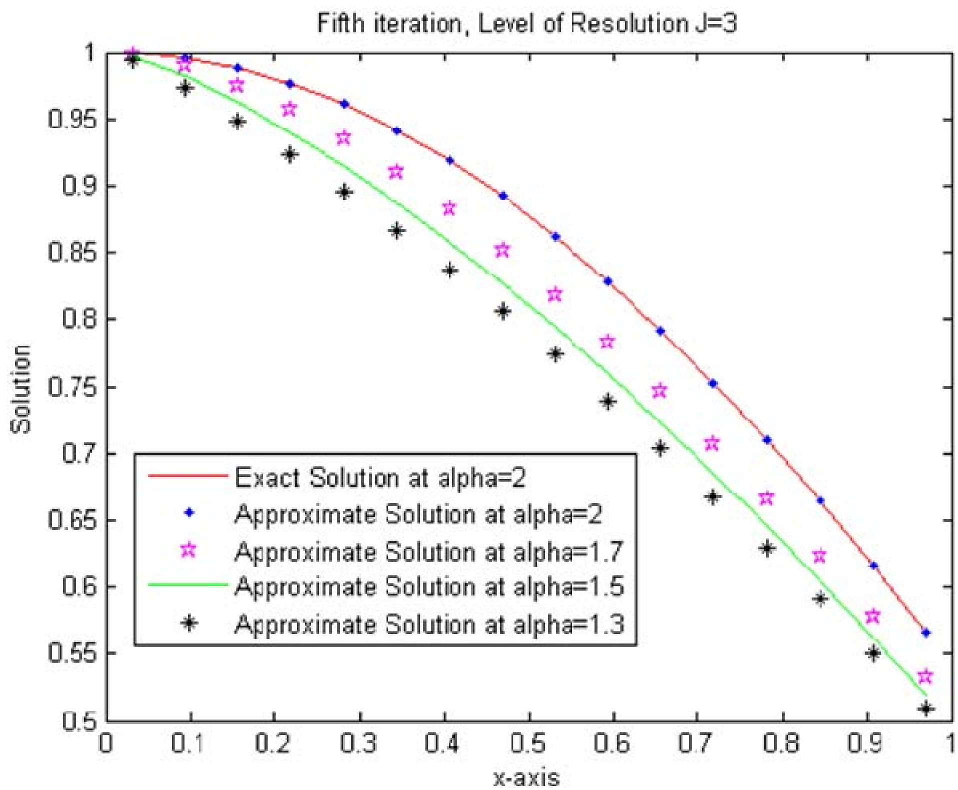

Figure 4. Exact solution at α = 2 and the Haar wavelet-Picard solution at α = 2, α = 1.8, α = 1.5, and α = 1.3.

Substituting Equations 24, 25 and 26 in Equation 23, we get

$$\sum_{i=1}^{2M} b_i \left[h_i(x) + \frac{2}{x} p_{\alpha-1,i}(x)\right] = -8e^{y_r(x)} - 4e^{\frac{y_r(x)}{2}}, \quad (27)$$

with the initial approximations $y_0(x) = 0, y'_0(x) = 0.$

Here we fix the order of differential Equation (21), $\alpha = 2$ and level of resolution, $J = 5$. The graph in Figure 5 shows the exact and approximate solutions by proposed method at six iterations. The absolute error reduces with increasing iterations.

Results of sixth iteration by Picard iteration at fixed level of resolution, $J = 5$, are shown in Figure 6 with exact solution at $\alpha = 2$ and the proposed solution at different values of α. Figure 6 showed that the proposed numerical solutions converge to the exact solution when α approaches to 2.

Example 4: Consider the α^{th} order fractional nonlinear boundary value problem,

$$^cD^{\alpha}y(x) + a(x)y'^2(x) + b(x)y(x)y'(x) = f(x),\ 1 < \alpha \leq 2, \quad (28)$$

subject to the boundary conditions $y(0) = 0, y(1) = 0.$

The exact solution is given by

$$y(x) = x^3 - x^2. \quad (29)$$

where
$f(x) = \frac{\Gamma(4)}{\Gamma(4-\alpha)} x^{3-\alpha} - \frac{\Gamma(3)}{\Gamma(3-\alpha)} x^{2-\alpha} + a(x)(3x^2 - 2x)^2 + b(x)(3x^2 - 2x)(x^3 - x^2).$

Applying the Picard technique to Equation (28), we get

$$^cD^{\alpha}y_{r+1}(x) = f(x) - a(x)y'^2_r(x) - b(x)y_r(x)y'_r(x),\ 1 < \alpha \leq 2, \quad (30)$$

with the boundary conditions $y_{r+1}(0) = 0, y_{r+1}(1) = 0.$

Now applying the Haar wavelet method to Equation (30), we approximate the higher order derivative term by the Haar wavelet series as

$$^cD^{\alpha}y_{r+1}(x) = \sum_{i=1}^{2M} b_i h_i(x) \quad (31)$$

Lower order derivatives are obtained by integrating Equation 31 and using the initial conditions

$$y_{r+1}(x) = \sum_{i=1}^{2M} b_i (p_{\alpha,i}(x) - xC_{\alpha,i}), \quad (32)$$

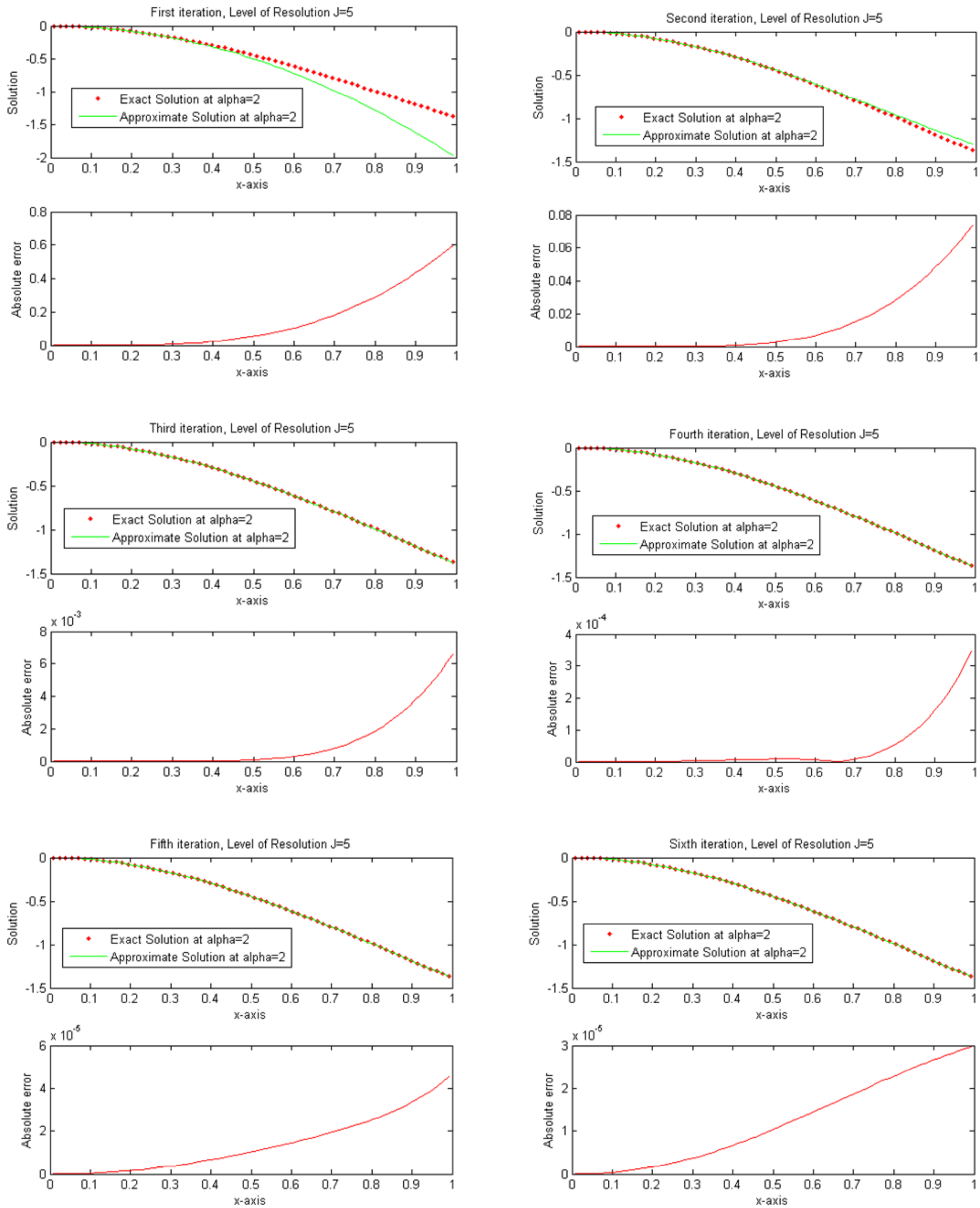

Figure 5. Comparison of exact solution and solutions by Haar wavelet-Picard technique at J = 5, for different iterations, and α = 2.

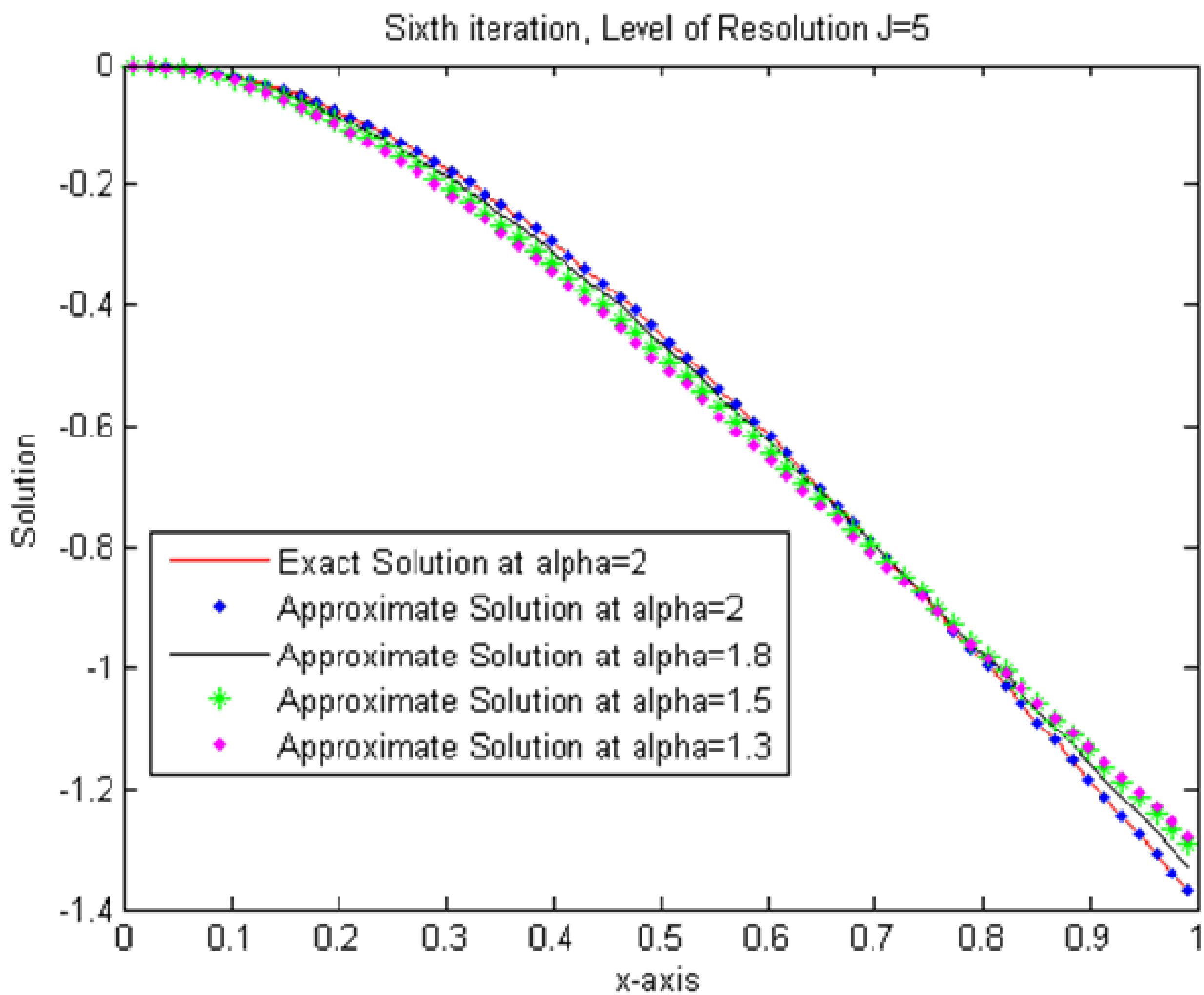

Figure 6. Exact solution at α = 2 and the Haar wavelet-Picard solution at α = 2, α = 1.8, α = 1.5, and α = 1.3.

$$y'_{r+1}(x) = \sum_{i=1}^{2M} b_i\left(p_{\alpha-1,i}(x) - xC_{\alpha,i}\right), \tag{33}$$

where $C_{\alpha,i} = \int_0^1 p_{\alpha,i}(x)\,dx$. Substituting Equations 31, 32 and 33 in Equation 30, we get

$$\sum_{i=1}^{2M} b_i h_i(x) = f(x) - a(x)y_r^2(x) - b(x)y_r(x)y'_r(x), \quad 1 < \alpha \leq 2 \tag{34}$$

with the initial approximation $y_0(x) = 0, y'_0(x) = 0$. Here we consider $a(x) = e^x$ and $b(x) = x$.

We fix the order of the differential Equation 28, $\alpha = 2$, and level of resolution, $J = 5$. The graph in Figure 7 shows the exact and approximate solutions by proposed method at six iterations. The absolute error reduces with increasing iterations.

Results at sixth iteration of proposed method at fixed level of resolution, $J = 5$, and at different values of α are shown in Figure 8 with the exact solution at $\alpha = 2$. Figure 8 showed that the numerical solutions converge to the exact solution when α approaches to 2.

Conclusion

This study showed that Haar wavelet-Picard technique gives excellent results when applied to different fractional order nonlinear initial and boundary value problems.

The solution of the fractional order, nonlinear differential equation converge to the solution of the integer order differential equation as shown in Figures 2, 4, 6 and 8.

Other Figures shows that approximate solution converge to the exact solution while iterations are increased and absolute error goes down.

Different type of nonlinearities can easily be handled by the Haar wavelet-Picard technique.

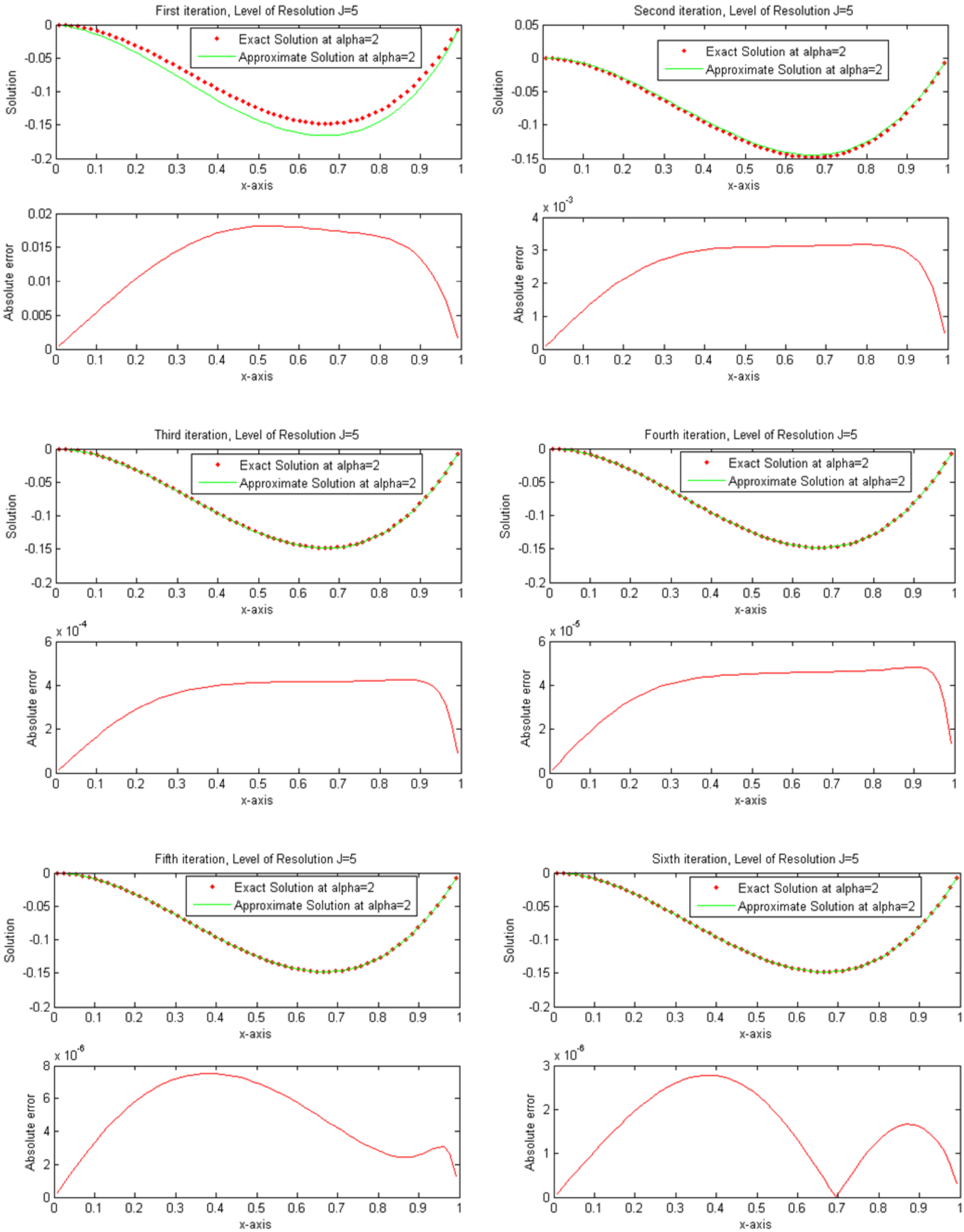

Figure 7. Comparison of exact solution and solutions by Haar wavelet-Picard technique at J = 5, for different iterations, and α = 2.

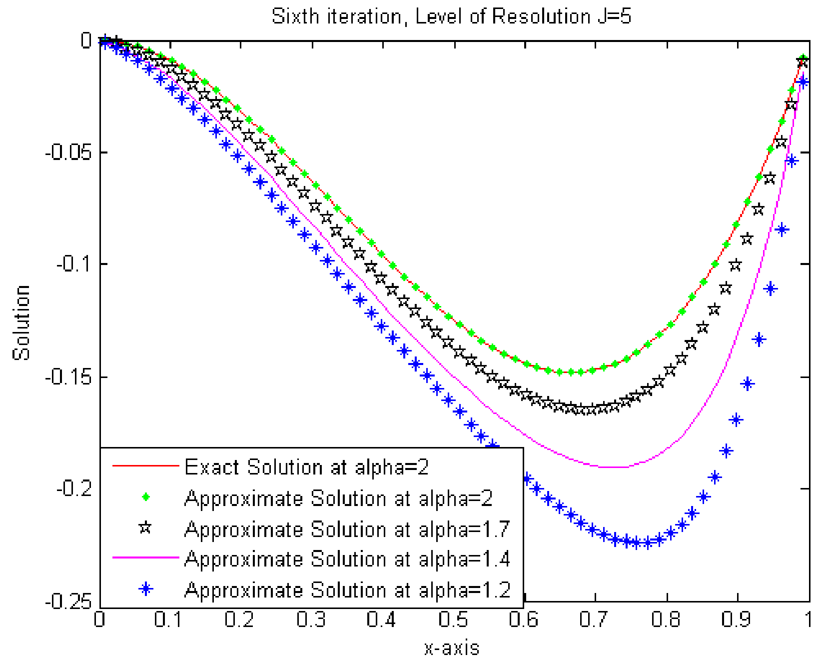

Figure 8. Exact solution at α = 2 and the Haar wavelet-Picard solution at α = 2, α = 1.7, α = 1.4, and α = 1.2.

Conflict of Interests

The author(s) have not declared any conflict of interests.

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