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Full Length Research Paper

Effects of curvature on free vibration characteristics of laminated composite cylindrical shallow shells

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This paper presents effects of curvature on free vibration characteristics of cross-ply laminated composite cylindrical shallow shells. Shallow shells have been considered for different lamination thickness, radius of curvature and elasticity ratio. First, kinematic relations of strains and deformation have been shown. Then, using Hamilton's principle, governing differential equations have been obtained for a general curved shell. In the next step, stress-strain relation for laminated, cross-ply composite shells has been given. By using some simplifications and assuming Fourier series as a displacement field, differential equations are solved by matrix algebra for shallow shells. The results obtained by this solution have been given in tables and graphs. The comparisons made with the literature and finite element program (ANSYS).

Key words: Structural composites, free vibration, effects of curvature, shear deformation shell theory, classical shell theory, finite element method (FEM).

INTRODUCTION

A composite is a structural material, which consists of combining two or more constituents on a macroscopic scale to form a useful material. The goal of this three dimensional composition is to obtain a property which none of the constituents possesses. In other words, the target is to produce a material that possesses higher performance properties than its constituent parts for a particular purpose. Some of these properties are mechanical strength, corrosion resistance, high tempera-ture resistance, heat conductibility, stiffness, lightness and appearance. In accordance with this definition, the following conditions must be satisfied by the composite material. It must be man-made and not natural. It must comprise at least two different materials with different chemical components separated by distinct interfaces. Different materials must be put together in a three dimensional unity. It must possess properties, which none of the constituents possesses alone and that must be the aim of its production. The material must behave as a whole, that is, the fiber and the matrix material (material surrounding the fibers) must be perfectly bonded. As a structural material, composites offer lower weight and higher strength

Shells are common structural elements in many engineering structures, including concrete roofs, exteriors of rockets, ship hulls, automobile tires, containers of liquids, oil tanks, pipes, aerospace etc. A shell can be defined as a curved, thin-walled structure. It can be made from a single layer or multilayer of isotropic or anisotropic materials. Shells can be classified according to their curvatures. Shallow shells are defined as shells that have rise of not more than one fifth the smallest planform dimension of the shell (Qatu, 2004).

Shells are three-dimensional (3D) bodies bounded by two relatively close, curved surfaces. The 3D equations of elasticity are complicated that is why all shell theories (thin, thick, shallow and deep, etc.) reduce the 3D elasticity problem into a 2D one. This is done usually by Classical Lamination Theory-CLT and Kirchhoff hypothesis.

A number of theories exist for layered shells. Many of these theories were developed originally for thin shells and based on the Kirchhoff–Love kinematic hypothesis that straight lines normal to the under formed mid-surface remain straight and normal to the middle surface after deformation. Among these theories Qatu (2004) uses

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energy functional to develop equation of motion. Many studies have been performed on characteristics of shallow shells (Qatu, 1991; Qatu, 1992; Qatu, 1993). Recently, Latifa and Sinha (2005) have used an improved finite element model for the bending and free vibration analysis of doubly curved, laminated composite shells having spherical and ellipsoidal shapes. Large-amplitude vibrations of circular cylindrical shells subjected to radial harmonic excitation in the spectral neighborhood of the lowest resonance are investigated by Amabili (2003). Gautham and Ganesan (1997) deal with the free vibration characteristics of isotropic and laminated orthotropic spherical caps. Liew et al. (2002) have presented the elasticity solutions for free vibration analysis of doubly curved shell panels of rectangular planform. Grigorenko and Yaremchenko (2007) have analyzed the stress-strain state of a shallow shell with rectangular planform and varying thickness. Djoudi and Bahai (2003) have presented a cylindrical strain based on shallow shell finite element which is developed for linear and geometrically non-linear analysis of cylindrical shells.

In this paper parameters affecting free vibration characteristic of symmetric, cross-ply, composite, shallow shells have been examined. The shells have square planform. The a/R (ratio of shell length to radius of shell) ratio has been considered as a parameter. For various a/R values solutions are obtained from computer program written using the following theory. Furthermore, for the same ratios, problem is modeled by finite element method also (Reddy, 1993). For the solution of problem by finite element method a commercial program, named ANSYS (ANSYS Inc), has been used. Starting from a/R=0 to 0.1 various values are examined by both computer program and ANSYS. Various a/h (ratio of shell length to thickness of shell) values are used as another parameter. The results obtained from analysis have been compared with literature and ANSYS by using tables and graphs.

Theories

A lamina is made of isotropic homogeneous reinforcing fibers and an isotropic homogeneous material surrounding the fibers, called matrix material (Figure 1). Therefore, the stiffness of the lamina varies from point to point depending on whether the point is in the fiber, the matrix or the fiber and matrix interface. Because of these variations, macro-mechanical analysis of a lamina is based on average properties.

There are many theories of shells. Classical shell theory, also known as Kirchhoff-Love kinematic hypo-thesis, assumes that “The normals to the middle surface remain straight and normal to the mid-surface when the shell undergoes deformation”. However, according to first order shear deformation theory “The transverse normals do not remain perpendicular to the mid-surface after deformation” (Reddy, 2003). In addition, classical lamination theory says “laminas are perfectly bonded”

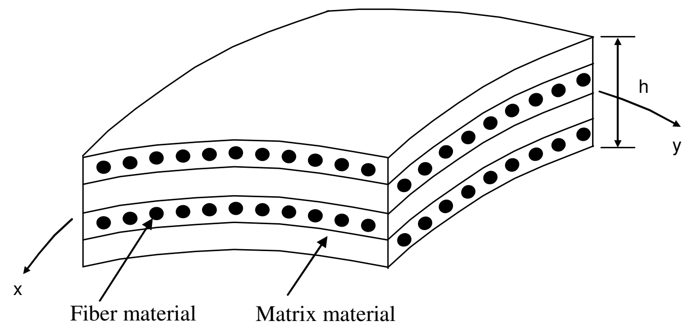


Figure 1. Fiber and matrix materials in laminated composite shallow shell.

(Gurdal and Haftka, 1998; Hyer, 1997; Reddy, 1995; Jones, 1984). The theory of shallow shells can be obtained by making the following additional assumptions to thin (or classical) and thick (or shear deformation) shell theories. It will be assumed that the deformation of the shells is completely determined by the displacement of its middle surface. The derivation of equations of motion is based on two assumptions. The first assumption is that the shallow shell has small deflections. The second assumption is that the shallow shell thickness is small compared to its radii of curvature. Also, the radii of curvature are very large compared to the in-plane displacement. Curvature changes caused by the tangential displacement component u and v are very small in a shallow shell, in comparison with changes caused by the normal component w.

Geometric properties

The vectorial equation of the undeformed surface could be written by the x and y cartesian coordinates as,

$$\bar{r} = \bar{r}(x, y) \tag{1}$$

a small increment in $\vec{r}$ vector is given as,

$$d\bar{r} = \bar{r}_{,x}\, dx + \bar{r}_{,y}\, dy \tag{2}$$

where $\bar{r}_{,x}$ is the small increment in x direction and $\bar{r}_{,y}$ is the small increment in y direction (Figure 2). The differential length of the shell surface could be found by dot product of $d\bar{r}$ by itself

$$ds^2 = d\bar{r}.d\bar{r} = A^2dx^2 + B^2dy^2 \tag{3}$$

where A and B are referred as Lame parameters and defined as

$$A^2 = \bar{r}_{,x}.\bar{r}_{,x} \quad , \qquad B^2 = \bar{r}_{,y}.\bar{r}_{,y} \tag{4}$$

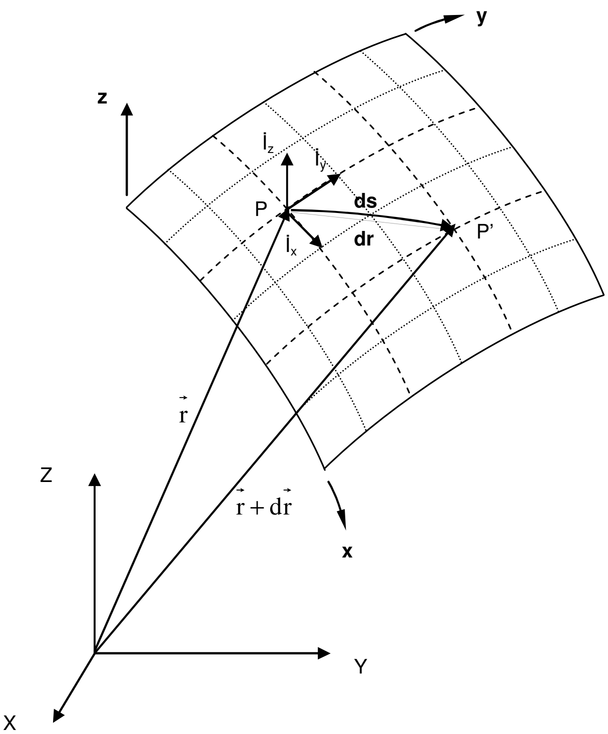


Figure 2. Coordinates of shell mid-surface.

Equation (3) is called first fundamental form of the surface. Tangent vector to the surface could be obtained by taking derivative of Equation (1) with respect to surface length. Then, applying Frenet's formula to the derivative of tangent vector and multiplying both sides by unit normal vector gives second quadratic form.

Kinematics of displacement

Let the position of a point, on a middle surface, be shown by $\bar{r}(x, y)$. If this point undergoes the displacement by the amount of $\overline{U}$ then, final position of that point could be given as,

$$\bar{r}'(x, y) = \bar{r}(x, y) + \overline{U} \tag{5}$$

where $\overline{U}$ is the displacement field of the point and defined as

$$\overline{U} = u\,\bar{i}_x + v\,\bar{i}_y + w\,\bar{i}_z \tag{6}$$

Where $\bar{i}_x$, $\bar{i}_y$ and $\bar{i}_z$ are the unit vectors in the direction of x, y and z. u, v, and w are the displacements in the direction of x, y and z respectively. Using Equations (5) and (6) strains are calculated as,

$$\begin{aligned}
\varepsilon_x &= \frac{1}{(1+z/R_x)}\left(\frac{1}{A}\frac{\partial u}{\partial x} + \frac{v}{AB}\frac{\partial A}{\partial y} + \frac{w}{R_x}\right)\\
\varepsilon_y &= \frac{1}{(1+z/R_y)}\left(\frac{1}{B}\frac{\partial v}{\partial y} + \frac{u}{AB}\frac{\partial B}{\partial x} + \frac{w}{R_y}\right)\\
\varepsilon_z &= \partial w/dz\\
\gamma_{xy} &= \frac{1}{(1+z/R_x)}\left(\frac{1}{A}\frac{\partial v}{\partial x} - \frac{u}{AB}\frac{\partial A}{\partial y} + \frac{w}{R_{xy}}\right) + \frac{1}{(1+z/R_y)}\left(\frac{1}{B}\frac{\partial u}{\partial y} - \frac{v}{AB}\frac{\partial B}{\partial x} + \frac{w}{R_{xy}}\right)\\
\gamma_{xz} &= \frac{1}{A(1+z/R_x)}\frac{\partial w}{\partial x} + A(1+z/R_x)\frac{\partial}{\partial z}\left(\frac{u}{A(1+z/R_x)}\right) - \frac{v}{R_{xy}(1+z/R_x)}\\
\gamma_{yz} &= \frac{1}{B(1+z/R_y)}\frac{\partial w}{\partial y} + B(1+z/R_y)\frac{\partial}{\partial z}\left(\frac{v}{B(1+z/R_y)}\right) - \frac{u}{R_{xy}(1+z/R_y)}
\end{aligned} \tag{7}$$

Where R_x, R_y, and R_{xy} are curvatures in x-plane, y-plane and xy- plane, respectively.

Stress-strain relation

For an orthotropic media there are 9 stiffness coefficients written in local coordinates.

$$[\sigma] = [Q][\varepsilon] \tag{8}$$

where $[\sigma]$ is the stress matrices, $[Q]$ is the stiffness matrices and $[\varepsilon]$ strain matrices. Equation (8) could be written in open form as,

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix}Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0\\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0\\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & Q_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & Q_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & Q_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{bmatrix} \tag{9}$$

Where:

$$Q_{11} = E_1\frac{1-\upsilon_{23}\upsilon_{32}}{\Delta},$$

$$Q_{12} = E_1\frac{\upsilon_{21}+\upsilon_{31}\upsilon_{23}}{\Delta} = E_2\frac{\upsilon_{12}+\upsilon_{32}\upsilon_{13}}{\Delta}$$

$$\begin{aligned}
Q_{22} &= E_2\frac{1-\upsilon_{31}\upsilon_{13}}{\Delta},\\
Q_{13} &= E_1\frac{\upsilon_{31}+\upsilon_{21}\upsilon_{32}}{\Delta} = E_3\frac{\upsilon_{13}+\upsilon_{12}\upsilon_{23}}{\Delta}\\
Q_{33} &= E_3\frac{1-\upsilon_{12}\upsilon_{21}}{\Delta},\\
Q_{23} &= E_2\frac{\upsilon_{32}+\upsilon_{12}\upsilon_{31}}{\Delta} = E_3\frac{\upsilon_{23}+\upsilon_{21}\upsilon_{13}}{\Delta}
\end{aligned} \tag{10}$$

$$Q_{44} = G_{23}, Q_{55} = G_{13}, Q_{66} = G_{12}$$
$$\Delta = 1 - \upsilon_{12}\upsilon_{21} - \upsilon_{23}\upsilon_{32} - \upsilon_{31}\upsilon_{13} - 2\upsilon_{21}\upsilon_{32}\upsilon_{13}$$

In Equation (10) subscribe 1 indicates fiber direction, subscribe 2 indicates matrix direction, subscribe 2 indicates direction that perpendicular to 1-2 plane. G is the shear modulus, E is the elasticity modulus and υ is the Poisson’s ratio. The stresses in global coordinates are calculated by applying transformation rules. Then, the stresses over the shell thickness are integrated to obtain the force and moment resultants. Due to curvatures of the structure, extra terms must be taken into account during the integration. This difficulty could be overcome by expanding the term $[1/(1+z/R_n)]$ in a geometric series.

Governing equations

Equation of motion for shell structures could be obtained by Hamilton’s principle

$$\delta\int_{t_1}^{t_2}(T + W - U)dt = 0 \qquad (11)$$

where T is the kinetic energy of the structure

$$T = \frac{\rho}{2}\int\left\{\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right\}dxdydz \qquad (12)$$

W is the work of the external forces

$$W = \int_x\int_y (q_x u + q_y v + q_z w + m_x\psi_x + m_y\psi_y)ABdxdy \qquad (13)$$

in which q_x, q_y, q_z are the external forces u, v, w are displacements in x, y, z direction respectively. m_x, m_y, are the external moments and ψ_x, ψ_y are rotations in x, y directions respectively. U is the strain energy defined as,

$$U = \frac{1}{2}\int(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \sigma_{xy}\varepsilon_{xy} + \sigma_{xz}\varepsilon_{xz} + \sigma_{yz}\varepsilon_{yz})dxdydz \qquad (14)$$

Solving Equation (11) gives set of equations called equations of motion for shell structures.

$$\frac{\partial}{\partial x}(BN_x) + \frac{\partial}{\partial y}(AN_{yx}) + \frac{\partial A}{\partial y}N_{xy} - \frac{\partial B}{\partial x}N_y + \frac{AB}{R_x}Q_x + \frac{AB}{R_{xy}}Q_y + ABq_x = AB(\bar{I}_1\ddot{u} + \bar{I}_2\ddot{\psi}_x)$$

$$\frac{\partial}{\partial y}(AN_y) + \frac{\partial}{\partial x}(BN_{xy}) + \frac{\partial B}{\partial x}N_{yx} - \frac{\partial A}{\partial y}N_x + \frac{AB}{R_y}Q_y + \frac{AB}{R_{xy}}Q_x + ABq_y = AB(\bar{I}_1\ddot{v} + \bar{I}_2\ddot{\psi}_y)$$

$$-AB\left(\frac{N_x}{R_x} + \frac{N_y}{R_y} + \frac{N_{xy} + N_{yx}}{R_{xy}}\right) + \frac{\partial}{\partial x}(BQ_x) + \frac{\partial}{\partial y}(AQ_y) + ABq_z = AB(\bar{I}_1\ddot{w}) \qquad (15)$$

$$\frac{\partial}{\partial x}(BM_x) + \frac{\partial}{\partial y}(AM_{yx}) + \frac{\partial A}{\partial y}M_{xy} - \frac{\partial B}{\partial x}M_y - ABQ_x + \frac{AB}{R_x}P_x + ABm_x = AB(\bar{I}_2\ddot{u} + \bar{I}_3\ddot{\psi}_x)$$

$$\frac{\partial}{\partial y}(AM_y) + \frac{\partial}{\partial x}(BM_{xy}) + \frac{\partial B}{\partial x}M_{yx} - \frac{\partial A}{\partial y}M_x - ABQ_y + \frac{AB}{R_y}P_y + ABm_y = AB(\bar{I}_2\ddot{v} + \bar{I}_3\ddot{\psi}_y)$$

When the shell has small curvature it is referred to as a shallow shell. Shallow shells are defined as shells that have a rise of not more than 1/5th the smallest planform dimension of the shell (Qatu, 2004). It has been widely accepted that shallow shell equations should not be used for maximum span to minimum radius ratio of 0.5 or more. For shallow shells, Lame parameters are assumed to equal to one (A=B=1). This gives Equation (15) in simplified form as,

$$\frac{\partial}{\partial x}N_x + \frac{\partial}{\partial y}N_{yx} + q_x = \bar{I}_1\ddot{u} + \bar{I}_2\ddot{\psi}_x$$

$$\frac{\partial}{\partial y}N_y + \frac{\partial}{\partial x}N_{xy} + q_y = \bar{I}_1\ddot{v} + \bar{I}_2\ddot{\psi}_y$$

$$-\left(\frac{N_x}{R_x} + \frac{N_y}{R_y} + \frac{N_{xy} + N_{yx}}{R_{xy}}\right) + \frac{\partial}{\partial x}Q_x + \frac{\partial}{\partial y}Q_y + q_z = \bar{I}_1\ddot{w} \qquad (16)$$

$$\frac{\partial}{\partial x}M_x + \frac{\partial}{\partial y}M_{yx} - Q_x + m_x = \bar{I}_2\ddot{u} + \bar{I}_3\ddot{\psi}_x$$

$$\frac{\partial}{\partial y}M_y + \frac{\partial}{\partial x}M_{xy} - Q_y + m_y = \bar{I}_2\ddot{v} + \bar{I}_3\ddot{\psi}_y$$

Equation (16) is defined as equation of motion for thick shallow shell. For thin shallow shells this equation reduces to,

$$\frac{\partial}{\partial x}N_x + \frac{\partial}{\partial y}N_{yx} + q_x = \bar{I}_1\ddot{u}$$

$$\frac{\partial}{\partial y}N_y + \frac{\partial}{\partial x}N_{xy} + q_y = \bar{I}_1\ddot{v} \qquad (17)$$

$$-\left(\frac{N_x}{R_x} + \frac{N_y}{R_y} + \frac{2N_{xy}}{R_{xy}}\right) + \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\partial y} + \frac{\partial^2 M_y}{\partial y^2} + q_z = \bar{I}_1\ddot{w}$$

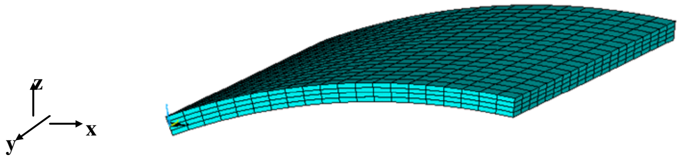


Figure 3. Cylindrical shallow shell.

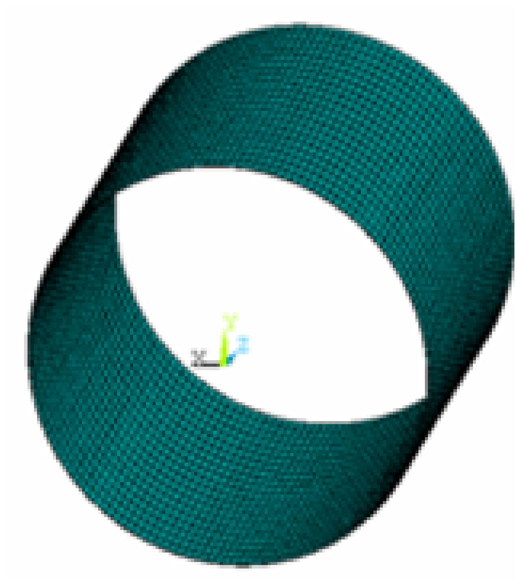


Figure 4. Cylindrical shells modeled by using ANSYS.

The Navier type solution can be applied to thick and thin shallow shells. This type solution assumes that the displacement field of the shallow shells could be represented as sine and cosine trigonometric functions.

Numerical examples

As an example a simply supported cylindrical shell which has a radius of curvature in one plane and infinite radius of curvature in other plane, is considered (Figure 3). The shell, in hand, has a square planform where length and width are equal to unity (a=b=1). As a material, a laminated composite has been used with a [0°/90°/90°/0°] symmetrical cross-ply stacking sequence. To determine the effect of shell thickness and radius of curvature on free vibration characteristics of cylindrical shallow shell, problem has been solved for various values. The ratio of modulus of elasticity (E_1/E_2) which is the ratio of modulus of elasticity in fiber direction to matrix direction has been taken as a variable from 1 to 50. Then, effect of shell thickness ratio that is ratio of shell width to shell thickness, a/h=100, 50, 20, 10 and 5, has been examined. Furthermore, radius of curvature has been considered. For different shell width/shell radius ratios which vary from infinity (plate) to 0.1, graphs have been obtained.

For each case, the shell has been solved with three theories. First theory is the classical laminated shallow shell theory (CLSST) which assumes "normals to the middle surface remain straight and normal after deformation". Second theory used in the solution of composite laminated shallow shell is shear deformation shallow shell theory (SDSST). SDSST is similar to CLSST except about transverse normals i.e., the trans-verse normals do not remain perpendicular to the mid-surface after deformation. The last theory is the Finite element model (FEM). Entire structure is meshed by finite elements in this theory. Then assuming a suitable displacement fields for each meshing element, the behavior of the structure has been obtained. In this paper, a finite element package program ANSYS has been used. The structure is meshed by 25×25 elements. A 8-noded quadratic element is considered as a meshing element named as SHELL99 (ANSYS Inc). The element has 100 layers to model the composite materials used in the structure. For each layer geometric and material properties is entered to program. Furthermore, thick-nesses of each layer, fiber orientations and stacking sequence must be entered carefully. During solution process, subspace and block Lanczos mode extracting methods are used separately to calculate first 30 frequencies.

Before proceeding further, the modeling of the shell structure in ANSYS package program has been checked to avoid getting wrong results. A cylindrical shell structure which is solved by Qatu (2004) as an example problem in section 7.3.1.1 has been chosen (Figure 4). The studies were made for isotropic steel. The thickness of the shell is h=0.02 in, the length of the shell is a=11.74 in, Radius of the cylindrical shell is R=5.836 in, unit mass is 734×10^{-6} lb s^2/in^4, modulus of elasticity is 29.5×10^6 lb/in^2 and Poisson's ratio is 0.285. The same cylindrical shell has been solved by Bert et al. (1993) using Love's shell theory, by Rat and Das who included shear deformation and rotary inertia; by Bray and Eagle using experimental procedure and finally by Qatu (2004) using classical shell theory. All the results obtained by researchers have been given in Qatu (2004). The same problem has been solved again by modeling the structure with finite element method and using ANSYS package program. A 160×20 mesh has been chosen. Each mesh element which is called SHELL99 has 8 degree of freedom. Results of that model prepared in ANSYS have been given in Table 1.

Using results given in Table 1 and results obtained by using ANSYS, the graphs have been drawn (Figure 5). The results have been given for first three (m=1, 2, 3) longitudinal modes and first thirty (n=1, 2,…, 30) circumferential modes. The three graphs have been drawn together in Figure 6.

The correctness of the ANSYS model has been checked in this example problem. The problem has been solved by Qatu (2004) Bert et al. (1993), Rath et al. (1973), Bray et al. (1970). The results obtained by those researchers have

Table 1. Natural frequency parameters (Hertz) obtained by using CLSST and ANSYS.

m	n	CLSST	ANSYS	m	n	CLSST	ANSYS	m	n	CLSST	ANSYS
	0	5328,25	5325,99		0	5442,58			0	5458,11	5446,97
	1	3270,54	3336,81		1	4837,71	4832,71		1	5197,96	5205,68
	2	1861,97	2144,79		2	3725,02	3729,93		2	4563,85	4565,18
	3	1101,78	1469,94		3	2742,67	2799,58		3	3813,65	3817,14
	4	705,71	1061,33		4	2018,09	2142,28		4	3114,51	3139,31
	5	497,54	803,13		5	1515,06	1684,79		5	2530,39	2587,26
	6	400,18	642,65		6	1174,98	1363,45		6	2069,45	2157,05
	7	380,82	556,52		7	953,72	1139,82		7	1719,25	1829,20
	8	416,82	533,12		8	824,39	993,07		8	1464,11	1585,77
	9	488,69	561,49		9	770,52	912,16		9	1291,20	1414,21
	10	583,96	628,94		10	778,47	889,54		10	1190,96	1306,51
	11	696,30	724,51		11	834,33	917,13		11	1154,97	1256,83
	12	822,76	840,92		12	925,62	985,64		12	1174,09	1259,24
	13	961,95	973,94		13	1043,11	1086,24		13	1238,37	1306,82
	14	1113,21	1121,24		14	1180,85	1211,93		14	1338,48	1392,15
1	15	1276,17	1281,56	2	15	1335,26	1357,80	3	15	1466,80	1508,31
	16	1450,68	1454,19		16	1504,23	1520,68		16	1617,78	1649,69
	17	1636,62	1638,73		17	1686,51	1698,52		17	1787,61	1812,07
	18	1833,93	1834,97		18	1881,37	1890,06		18	1973,77	1992,46
	19	2042,59	2042,76		19	2088,34	2094,50		19	2174,60	2188,81
	20	2262,58	2262,04		20	2307,14	2311,34		20	2389,01	2399,71
	21	2493,87	2492,77		21	2537,59	2540,26		21	2616,30	2624,22
	22	2736,47	2734,93		22	2779,58	2781,06		22	2855,96	2861,71
	23	2990,36	2988,52		23	3033,04	3033,61		23	3107,69	3111,75
	24	3255,54	3253,55		24	3297,90	3297,84		24	3371,26	3374,05
	25	3532,02	3529,36		25	3574,13	3573,71		25	3646,52	3648,43
	26	3819,79	3818,06		26	3861,71	3861,21		26	3933,35	3934,75
	27	4118,84	4117,60		27	4160,63	4160,34		27	4231,69	4232,96
	28	4429,18	4428,72		28	4470,86	4471,14		28	4541,48	4543,02
	29	4750,81	4751,49		29	4792,41	4793,63		29	4862,68	4864,92
	30	5083,73	5085,96		30	5125,26	5127,88		30	5195,25	5198,69

been compared by the results obtained by modeling the problem in ANSYS. For three cases graphs have been drawn and a perfect match has been observed with the results. This proofs the correctness of the model entered in ANSYS.

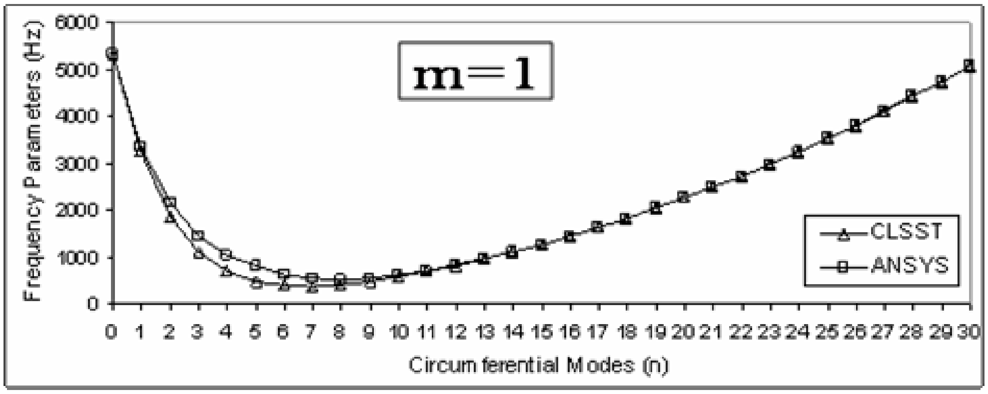
The governing Equation (16) (using SDSST theory) and the governing Equation (17) (using CLSST theory) derived in the theory section are solved by using Mathematical program separately. Furthermore, ANSYS packet program has been used in the solution. The geometry of the shell structures has been created using arc-length method in ANSYS. Then, area element has been defined between the arc lines. Finally using SHELL 99 finite element, the area has been meshed.

The problem defined at the beginning of this section has been solved by ANSYS and Mathematical program (Figure 3). The results obtained by ANSYS and Mathematical have been compared in tables and graphs.

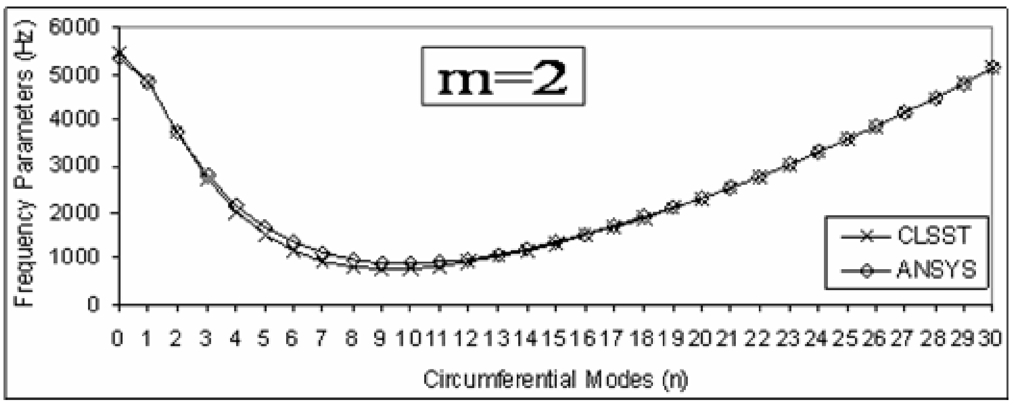
Tables 2, 3, 4 and 5 give non-dimensional natural frequency parameters,

$$(\Omega = \omega a^2 \sqrt{\rho / E_2 h^2})$$

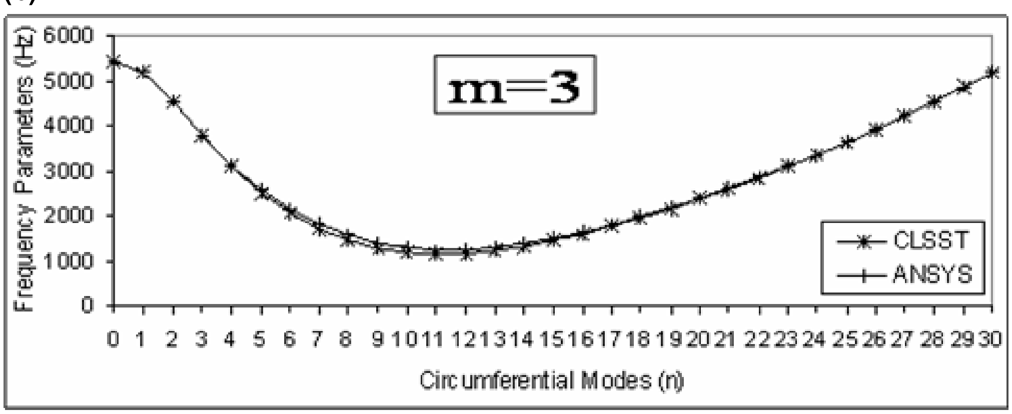
varying with shell thickness, shell curvature and shell anisotropy. The planform dimensions of the shell are equal to unity. For each case, three solutions have been carried out. Cylindrical shallow shells have been solved by Mathematical program with the shear deformation shallow shell theory (SDSST) and classical shallow shell theory (CLSST). The results obtained by using both theories are the same given by Qatu (2004). However, ANSYS results have differed from the other results. Figures 7 and 8 show variation of natural frequency parameters ($\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$) versus elasticity ratio effect



(a)



(b)



(c)

Figure 5. Comparison of the results given by using CLSST and ANSYS.

effect, shell thickness ratio effect and shell curvature effect for cross ply symmetrically laminated [0º/90º/90º/0º] cylindrical shallow shells. The graphs have been drawn according to results obtained by three theory; shear deformation shallow shell theory (SDSST), classical shallow shell theory (CLSST) and finite element method.

Conclusions

In this study, free vibration characteristic of symmetrically laminated, cylindrical, composite shallow shells have been investigated by using three different theories. Effects of shell curvatures and shell thicknesses have been shown with various graphs and tables for shallow shells which have square planform. The tables give non-dimensional natural frequencies versus shell thickness and shell curvatures using three different theories. Each table has been prepared for different material anisotropy value (E_1/E_2). Analysis and assumptions used in the SDSST and CLSST is similar to that used in Qatu (2004).

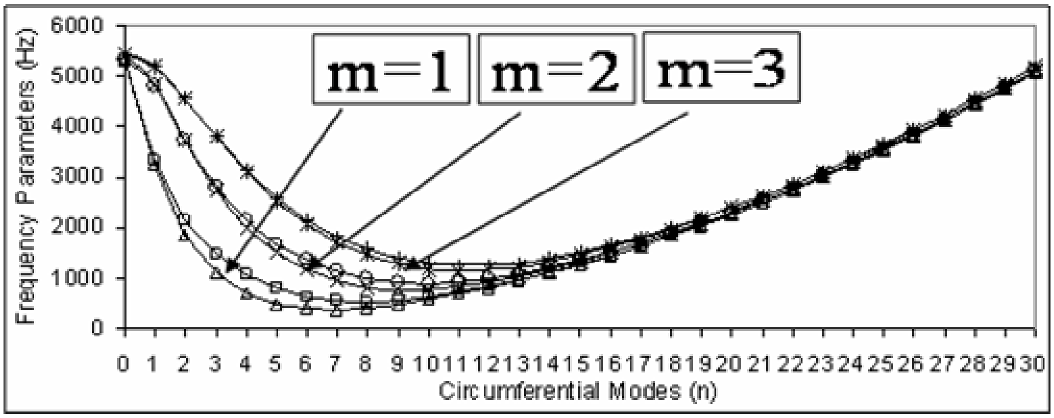


Figure 6. The results for first three (m=1, 2, 3) longitudinal modes.

Finite element analysis has been performed by using commercial finite element program named ANSYS.

In the tables, the following results have been observed. The curvature of shallow shells has the increasing effects on the non-dimensional natural frequencies. As the curvature value increases the non-dimensional natural frequencies also increase. Furthermore, as the curvature value increases the non-dimensional natural frequencies obtained by the solutions of the three theories differ from each other. These differences are mainly caused by the different assumptions between the theories. Next, the thickness effect has been studied. The first important result gained from tables is that, as the thickness increases the results from CLSST (thin shell) differ from other two theories, as expected. For cylindrical shallow shells, shell thickness has the effect on the results of the three theories. That is, the results of the theories get closer as the shell thickness varies from thin to thick. For example, in Table 5 for a/h=100, a/R=0.01 values, the results are 49.8238221, 21.8869447 and 21.9651424 from ANSYS solution, from SDSST solution and from CLSST solution, respectively. Similarly for a/h=50, a/R=0.01 values, the results are 30.6143513, 20.8100439 and 21.1282430 from ANSYS solution, from SDSST solution and from CLSST solution, respectively. The last observation for the thickness is about the rate of change of the non-dimensional natural frequencies. The non-dimensional natural frequency for a shallow shell in Table 5, for example, varies from 20.76 to 49.82 for a shell thickness ratio 100, and from 20.51 to 30.61 for a shell thickness ratio 50 and finally, from 19.07 to 21.13 for a shell thickness ratio 20. This means that for the first case the rate of change is 140%; for the second case the rate of change is 44% and for the last case the rate of change is 10.8%. The last observation on tables gives the anisotropy effect. Different material anisotropy values have been considered for each table. A careful examination between tables shows the increase in anisotropy causes increase in the non-dimensional natural frequency values. In addition, this increase also causes the results of ANSYS differ from others.

In Figure 7, frequency parameter versus curvature ratio graphs have been drawn. Solution of three methods has been shown on each graph. Two anisotropy cases have been considered $E_1/E_2=1$ on the left hand side and 15 on the right hand side. Furthermore, graphs in the each line in Figure 7 have been drawn for the same thickness ratio. The important point must be noticed here. As the curvature ratio increases, the results of ANSYS and other two methods differ. At the left hand side of Figure 7, this difference stars at where the curvature ratio is approximately 0.03 for the shallow shell $E_1/E_2=1$ and a/h=100, at the right hand side of Figure 7, this difference stars at where the curvature ratio is approximately 0.01 for the shallow shell $E_1/E_2=15$ and a/h=100. It is con- cluded that assumption of lame parameters equals to unity for shallow shells in the analysis of CLSST and SDSST gets fail as the curvature ratio increase. But luckily, the effect of this assumption decreases as the shell gets thicker. In Figure 8, frequency parameter versus elasticity ratio graphs have been drawn for different curvature ratios. The solutions obtained by different theories have been given on a separate graph. Graphs in each line of Figure 8 have been drawn for the same thickness ratio. The differences in frequency parameters of ANSYS solution for the each case of curvature ratios have been represented better than other two theories. According to shell thickness whether shell is thick or thin, the frequency parameter values also differ in ANSYS solution vice versa the solution of other theories has not been much affected due to shell thickness. The last conclusion must be mentioned here. The results of SDSST for the thick shallow shell case have coincided with ANSYS solutions but differ with CLSST solutions as expected.

In the literature shallow shells are defined as “shells that have rise of not more than one fifth the smallest planform dimension of the shell”. In this study, to verify shallow shell definition, FEM solutions have been compared

Table 2. Natural frequency parameters ($\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$) for cross ply laminated [0º/90º/90º/0º] cylindrical shallow shells for shear deformation shallow shell theory (SDSST), classical shallow shell theory (CLSST) and ANSYS. ($a/b = 1$, $E_1/E_2 = 5$, $G_{12}/E_2 = G_{13}/E_2 = G_{13}/E_2 = 0.5$ $\upsilon_{12} = 0.25$ and $K^2 = 5/6$).

a/h	a/R	ANSYS	SDSST	CLSST
100	0.000	8.3182320	8.3419561	8.3466158
	0.005	8.3554634	8.3476910	8.3523485
	0.010	8.4660912	8.3648721	8.3695228
	0.020	8.8945184	8.4332445	8.4378686
	0.025	9.2028989	8.4841601	8.4887647
	0.033	9.8354765	8.5931268	8.5976906
	0.050	11.4494426	8.8970644	8.9015209
	0.100	17.7495839	10.3847602	10.3888021
50	0.000	8.2824179	8.3280318	8.3466158
	0.005	8.2918191	8.3294614	8.3480430
	0.010	8.3198537	8.3337484	8.3523233
	0.020	8.4309702	8.3508743	8.3694217
	0.025	8.5132524	8.3636951	8.3822221
	0.033	8.6884797	8.3913259	8.4098092
	0.050	9.1705102	8.4697626	8.4881238
	0.100	11.4231408	8.8810714	8.8988334
20	0.000	8.1342034	8.2328074	8.3466158
	0.005	8.1357984	8.2330311	8.3468371
	0.010	8.1403834	8.2337022	8.3475009
	0.020	8.1586703	8.2363860	8.3501554
	0.025	8.1722922	8.2383982	8.3521457
	0.033	8.2017396	8.2427437	8.3564439
	0.050	8.2852836	8.2551448	8.3687102
	0.100	8.7224188	8.3217438	8.4345926
10	0.000	7.7777913	7.9214445	8.3466158
	0.005	7.7783773	7.9214963	8.3466647
	0.010	7.7795769	7.9216516	8.3468112
	0.020	7.7843752	7.9222728	8.3473973
	0.025	7.7879295	7.9227386	8.3478368
	0.033	7.7956157	7.9237449	8.3487863
	0.050	7.8176524	7.9266189	8.3514980
	0.100	7.9353888	7.9421093	8.3661152
5	0.000	6.8597888	6.9862196	8.3466158
	0.005	6.8603666	6.9862292	8.3466210
	0.010	6.8607220	6.9862580	8.3466364
	0.020	6.8620771	6.9863732	8.3466982
	0.025	6.8630546	6.9864596	8.3467445
	0.033	6.8652316	6.9866463	8.3468446
	0.050	6.8714516	6.9871794	8.3471304
	0.100	6.9048843	6.9900556	8.3486726

Table 3. Natural frequency parameters ($\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$) for cross ply laminated [0º/90º/90º/0º] cylindrical shallow shells for shear deformation shallow shell theory (SDSST), classical shallow shell theory (CLSST) and ANSYS. ($a/b = 1$, $E_1/E_2 = 15$, $G_{12}/E_2 = G_{13}/E_2 = G_{13}/E_2 = 0.5$ $\upsilon_{12} = 0.25$ and $K^2 = 5/6$).

a/h	a/R	ANSYS	SDSST	CLSST
100	0.000	12.2459182	12.2614750	12.2773269
	0.005	12.3124726	12.2660077	12.2818545
	0.010	12.5098255	12.2795957	12.2954271
	0.020	13.2687588	12.3337962	12.3495665
	0.025	13.8119456	12.3742888	12.3900138
	0.033	14.9156466	12.4613180	12.4769468
	0.050	17.6894273	12.7066474	12.7220125
	0.100	28.2344322	13.9561431	13.9703189
50	0.000	12.1857172	12.2143399	12.2773269
	0.005	12.2026001	12.2154667	12.2784485
	0.010	12.2526936	12.2188463	12.2818126
	0.020	12.4510017	12.2323552	12.2952591
	0.025	12.5975724	12.2424766	12.3053339
	0.033	12.9086409	12.2643138	12.3270710
	0.050	13.7583422	12.3264822	12.3889561
	0.100	17.6461314	12.6566525	12.7176703
20	0.000	11.8491688	11.9009970	12.2773269
	0.005	11.8520122	11.9011702	12.2774946
	0.010	11.8602760	11.9016900	12.2779976
	0.020	11.8932422	11.9037685	12.2800092
	0.025	11.9178558	11.9053271	12.2815177
	0.033	11.9709038	11.9086935	12.2847758
	0.050	12.1213398	11.9183050	12.2940783
	0.100	12.9029318	11.9700316	12.3441502
10	0.000	10.9214948	10.9716272	12.2773269
	0.005	10.9223390	10.9716650	12.2773581
	0.010	10.9246049	10.9717786	12.2774517
	0.020	10.9335351	10.9722330	12.2778261
	0.025	10.9401994	10.9725737	12.2781069
	0.033	10.9546387	10.9733098	12.2787134
	0.050	10.9957798	10.9754123	12.2804458
	0.100	11.2153027	10.9867495	12.2897888
5	0.000	8.7742495	8.7784062	12.2773269
	0.005	8.7743828	8.7784119	12.2773232
	0.010	8.7750937	8.7784288	12.2773121
	0.020	8.7778705	8.7784966	12.2772676
	0.025	8.7799142	8.7785475	12.2772343
	0.033	8.7844015	8.7786574	12.2771622
	0.050	8.7971748	8.7789713	12.2769563
	0.100	8.8658173	8.7806649	12.2758456

Table 4. Natural frequency parameters ($\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$) for cross ply laminated [0º/90º/90º/0º] cylindrical shallow shells for shear deformation shallow shell theory (SDSST), classical shallow shell theory (CLSST) and ANSYS. ($a/b = 1$, $E_1/E_2 = 25$, $G_{12}/E_2 = G_{13}/E_2 = G_{13}/E_2 = 0.5$ $\upsilon_{12} = 0.25$ and $K^2 = 5/6$).

a/h	a/R	ANSYS	SDSST	CLSST
100	0.000	15.1848853	15.1966195	15.2277943
	0.005	15.2714771	15.2004253	15.2315928
	0.010	15.5291199	15.2118367	15.2429825
	0.020	16.5179723	15.2573956	15.2884551
	0.025	17.2247461	15.2914739	15.3224693
	0.033	18.6565540	15.3648371	15.3956952
	0.050	22.2396947	15.5725053	15.6029822
	0.100	35.7649855	16.6483393	16.6769989
50	0.000	15.0831433	15.1043949	15.2277943
	0.005	15.1052244	15.1053385	15.2287305
	0.010	15.1708014	15.1081688	15.2315388
	0.020	15.4301547	15.1194844	15.2427663
	0.025	15.6217762	15.1279651	15.2511812
	0.033	16.0279668	15.1462701	15.2693443
	0.050	17.1349999	15.1984395	15.3211112
	0.100	22.1675867	15.4768842	15.5974535
20	0.000	14.4766898	14.5084431	15.2277943
	0.005	14.4808661	14.5085857	15.2279289
	0.010	14.4918844	14.5090135	15.2283327
	0.020	14.5356024	14.5107244	15.2299479
	0.025	14.5682132	14.5120074	15.2311591
	0.033	14.6385884	14.5147787	15.2337755
	0.050	14.8377184	14.5226923	15.2412468
	0.100	15.8694447	14.5653161	15.2814952
10	0.000	12.8759190	12.8912312	15.2277943
	0.005	12.8778295	12.8912607	15.2278141
	0.010	12.8808951	12.8913493	15.2278737
	0.020	12.8932463	12.8917036	15.2281121
	0.025	12.9023542	12.8919693	15.2282909
	0.033	12.9222139	12.8925433	15.2286771
	0.050	12.9787274	12.8941828	15.2297803
	0.100	13.2795994	12.9030251	15.2357311
5	0.000	9.7176957	9.6925563	15.2277943
	0.005	9.7194062	9.6925597	15.2277845
	0.010	9.7204281	9.6925699	15.2277551
	0.020	9.7244933	9.6926106	15.2277551
	0.025	9.7274923	9.6926412	15.2277551
	0.033	9.7340677	9.6927071	15.2277551
	0.050	9.7527723	9.6928956	15.2277551
	0.100	9.8531592	9.6939123	15.2277551

Table 5. Natural frequency parameters ($\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$) for cross ply laminated [0º/90º/90º/0º] cylindrical shallow shells for shear deformation shallow shell theory (SDSST), classical shallow shell theory (CLSST) and ANSYS. ($a/b = 1$, $E_1/E_2 = 50$, $G_{12}/E_2 = G_{13}/E_2 = G_{13}/E_2 = 0.5$ $\upsilon_{12} = 0.25$ and $K^2 = 5/6$).

a/h	a/R	ANSYS	SDSST	CLSST
100	0.000	20.7615920	20.7697041	20.8519882
	0.005	20.8861260	20.7725750	20.8548479
	0.010	21.2546187	20.7811850	20.8634248
	0.020	22.6662115	20.8155881	20.8976956
	0.025	23.6739017	20.8413518	20.9233606
	0.033	25.7098528	20.8969019	20.9786985
	0.050	30.7856245	21.0547801	21.1359801
	0.100	49.8238221	21.8869447	21.9651424
50	0.000	20.5170113	20.5293619	20.8519882
	0.005	20.5487779	20.5300691	20.8526841
	0.010	20.6430781	20.5321904	20.8547715
	0.020	21.0158582	20.5406734	20.8631190
	0.025	21.2911836	20.5470330	20.8693772
	0.033	21.8740232	20.5607652	20.8828904
	0.050	23.4588662	20.5999425	20.9214446
	0.100	30.6143513	20.8100439	21.1282430
20	0.000	19.0690313	19.0768068	20.8519882
	0.005	19.0751625	19.0769093	20.8520780
	0.010	19.0914234	19.0772169	20.8523476
	0.020	19.1564672	19.0784473	20.8534259
	0.025	19.2048946	19.0793699	20.8542346
	0.033	19.3093024	19.0813631	20.8559814
	0.050	19.6045764	19.0870555	20.8609706
	0.100	21.1261749	19.1177388	20.8878684
10	0.000	15.8118649	15.7948517	20.8519882
	0.005	15.8149750	15.7948703	20.8519912
	0.010	15.8198621	15.7949262	20.8520001
	0.020	15.8394997	15.7951499	20.8520358
	0.025	15.8540723	15.7953176	20.8520625
	0.033	15.8857057	15.7956800	20.8521203
	0.050	15.9756740	15.7967152	20.8522854
	0.100	16.4526620	15.8022988	20.8531763
5	0.000	10.9302473	10.8910270	20.8519882
	0.005	10.9327131	10.8910274	20.8519681
	0.010	10.9344903	10.8910288	20.8519079
	0.020	10.9415767	10.8910341	20.8516669
	0.025	10.9468415	10.8910382	20.8514862
	0.033	10.9582819	10.8910469	20.8510959
	0.050	10.9908705	10.8910719	20.8499807
	0.100	11.1651203	10.8912067	20.8439639

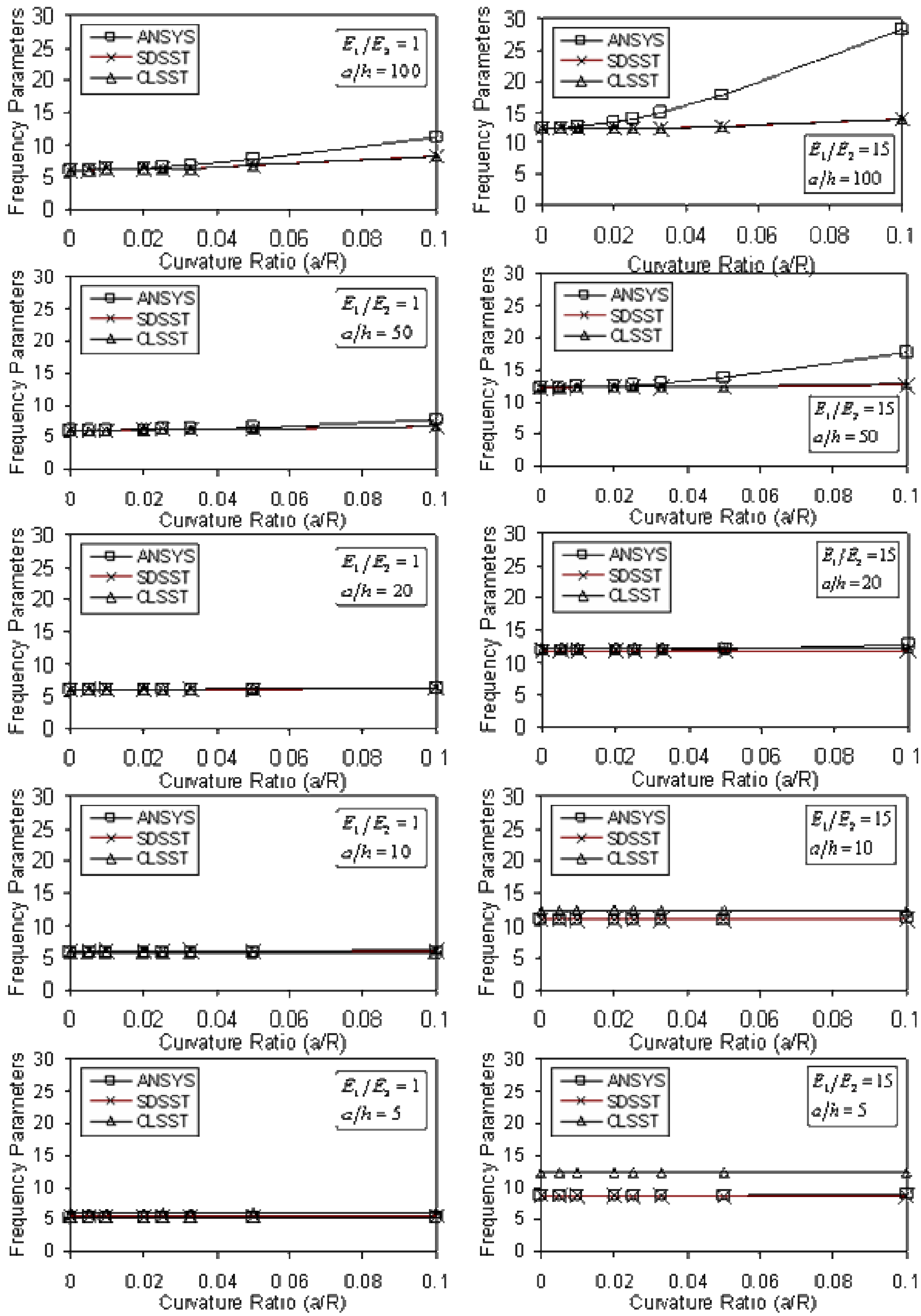


Figure 7. Natural frequency parameters ($\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$) and curvature effect for cross ply laminated [0º/90º/90º/0º] cylindrical shallow shells for shear deformation shallow shell theory (SDSST), classical shallow shell theory (CLSST) and ANSYS.

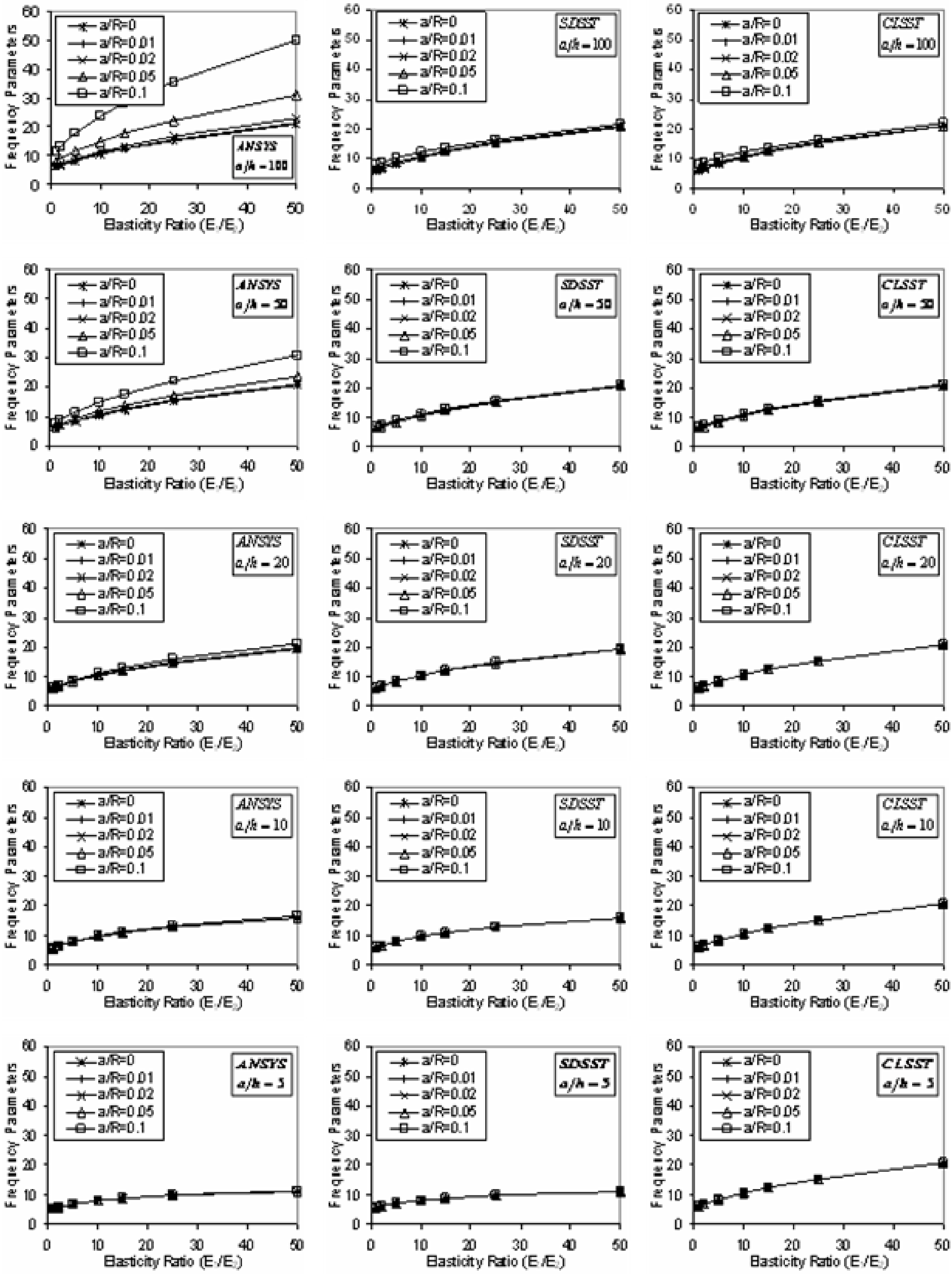


Figure 8. Natural frequency parameters ($\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$) and elasticity ratio effect for cross ply laminated [0º/90º/90º/0º] cylindrical shallow shells for shear deformation shallow shell theory (SDSST), classical shallow shell theory (CLSST) and ANSYS.

with CLSST and SDSST solutions, which assume Lame parameters equal to one (A=B=1), for different situations. As the thickness of the shallow shell increases, the results of FEM and other two theories get closer. This statement could be explained as follows. For thin shallow shells, FEM results starts to differ from other theories at the 0.03 curvature ratio whereas solutions of three coincides even at the 0.1 curvature ratio. Elasticity ratio that is, anisotropy, also affects the results. For isotropic case results of three theories agree with each other. However, as the anisotropy increases results of the theories gets differ.

As a conclusion, it could be said that, for shallow shells, no general definition could be done without considering effects of curvature ratio and thickness ratio.

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