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# A note on absolute summability factor theorem and almost increasing sequences

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**Quite recently Savas (2008) has proved a theorem on $|A|_k$ - summability factors of an infinite series. The present paper deals with a further generalization of it.**

**Key words:** Absolute summability, almost increasing, weighted mean matrix, summability factor.

## INTRODUCTION

Savas (2008) obtained sufficient conditions for $\sum a_n \lambda_n$ to be summable $|A|_k, k \geq 1$. In this paper a theorem on $|A,\delta|_k$-summability methods has been proved. A sequence $(b_n)$ of positive numbers is said to be $\delta$-quasi-monotone, if $b_n > 0$ ultimately and $\Delta b_n \geq -\delta_n$, where $(\delta_n)$ is a sequence of positive numbers (Savas, 2008). Let A be a lower triangular matrix, $\{s_n\}$ a sequence. Then,

$$A_n := \sum_{v=0}^{n} a_{nv} s_v .$$

A series $\sum a_n$ is said to be summable $|A|_k, k \geq 1$ if;

$$\sum_{n=1}^{\infty} n^{k-1} | A_n - A_{n-1} |^k < \infty, \tag{1}$$

and it is said to be summable $|A,\delta|_k$, $k \geq 1$ and $\delta \geq 0$ if (Flett, 1957):

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$$\sum_{n=1}^{\infty} n^{\delta k+k-1} |A_n - A_{n-1}|^k < \infty. \tag{2}$$

We may associate with A, two lower triangular matrices $\bar{A}$ and $\hat{A}$ defined as follows:

$$\bar{a}_{n,v} := \sum_{r=v}^{n} a_{nr}, \quad n, v = 0,1,2,... \quad \text{and}$$

$$\hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v} \qquad n = 1,2,3,....$$

A triangle is a lower triangular matrix with all non-zero main diagonal entries. A positive sequence $\{d_n\}$ is said to be almost increasing if there exist a positive increasing sequence $\{c_n\}$ and two positive constants $A$ and $B$ such

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that $Ac_n \le d_n \le Bc_n$ for each n.

## RESULTS

We have the following theorem:

**Theorem 1.** Let A be a lower triangular matrix with non-negative entries satisfying:

i) $\bar{a}_{n0} = 1,$

ii) $a_{n-1,v} \ge a_{nv}$ for , $n \ge v+1$

iii) $na_{nn} = O(1),$

iv) $\sum_{n=v+1}^{m+1} n^{\delta k} \mid \Delta_v \hat{a}_{nv} \mid = \mathrm{O}(v^{\delta k} a_{vv})$

v) $\sum_{n=v+1}^{m+1} n^{\delta k} \mid \hat{a}_{nv+1} \mid = \mathrm{O}(v^{\delta k})$

If $\{X_n\}$ is an almost increasing sequence such that, $|\Delta X_n| = O\left(\frac{X_n}{n}\right)$ and $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers $(A_n)$ such that it is $\delta$-quasi monotone with $\sum nX_n\delta_n < \infty$, $\sum A_n X_n$ is convergent and $|\Delta\lambda_n| \le |A_n|$ for all *n*. If;

vi) $\sum_{n=1}^{\infty} \frac{|\lambda_n|}{n} < \infty$

vii) $\sum_{n=1}^{m} n^{\delta k-1} \mid t_n \mid^k = \mathrm{O}(X_m), \quad where\ t_n := \frac{1}{n+1} \sum_{k=1}^{n} ka_k,$

then the series $\sum a_n \lambda_n$ is summable $|A,\delta|_k$, $k \ge 1$, $\delta \ge 0$.

We need the following lemmas for the proof of Theorem 1.

**Lemma 1.** Under the conditions of the theorem (Savas, 2008), we have that:

(1) $|\lambda_n| X_n = O(1).$

**Lemma 2.** Let $\{X_n\}$ is an almost increasing sequence such that $|\Delta X_n| = O\left(\frac{X_n}{n}\right)$ (Savas, 2008). If $(A_n)$ is $\delta$-quasi monotone with $\sum nX_n\delta_n < \infty$, $\sum A_n X_n$ is convergent, then,

(2) $\sum_{n=1}^{\infty} nX_n |\Delta A_n| < \infty,$ *and*

(3) $nA_n X_n = O(1).$

***Proof***:

Let $\{y_n\}$ be the nth term of the A-transform of $\sum_{i=0}^{n} \lambda_i a_i$. Then,

$$y_n := \sum_{i=0}^{n} a_{ni} s_i = \sum_{i=0}^{n} a_{ni} \sum_{v=0}^{i} \lambda_v a_v$$
$$= \sum_{v=0}^{n} \lambda_v a_v \sum_{i=v}^{n} a_{ni} = \sum_{v=0}^{n} \bar{a}_{nv} \lambda_v a_v$$

and

$$Y_n := y_n - y_{n-1} = \sum_{v=0}^{n} \left(\bar{a}_{nv} - \bar{a}_{n-1,v}\right) \lambda_v a_v = \sum_{v=0}^{n} \hat{a}_{nv} \lambda_v a_v \qquad (3)$$

We may write:

$$Y_n = \sum_{v=1}^{n} \left(\frac{\hat{a}_{nv}\lambda_v}{v}\right) v a_v$$
$$= \sum_{v=1}^{n} \left(\frac{\hat{a}_{nv}\lambda_v}{v}\right) \left[\sum_{r=1}^{v} r a_r - \sum_{r=1}^{v-1} r a_r\right]$$
$$= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv}\lambda_v}{v}\right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{v=1}^{n} v a_v$$

$$= \sum_{v=1}^{n-1} \left(\Delta_v \hat{a}_{nv}\right) \lambda_v \frac{v+1}{v} t_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \left(\Delta\lambda_v\right) \frac{v+1}{v} t_v$$
$$+ \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \frac{1}{v} t_v + \frac{(n+1) a_{nn} \lambda_n t_n}{n}$$
$$= T_{n1} + T_{n2} + T_{n3} + T_{n4}, \text{ say.}$$

To complete the proof it is sufficient, by Minkowski's inequality, to show that:

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}\left|T_{nr}\right|^k < \infty, \quad for\ r = 1,\ 2,\ 3,\ 4.$$

Using Hölder's inequality and (iii),

$$I_1 := \sum_{n=1}^{m} n^{\delta k+k-1}\left|T_{n1}\right|^k = \sum_{n=1}^{m} n^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\Delta_v \hat{a}_{nv}\lambda_v \frac{v+1}{v}t_v\right|^k$$

$$= O(1)\sum_{n=1}^{m+1} n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}\left|\Delta_v \hat{a}_{nv}\right|\left|\lambda_v\right|\left|t_v\right|\right)^k$$

$$= O(1)\sum_{n=1}^{m+1} n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}\left|\Delta_v \hat{a}_{nv}\right|\left|\lambda_v\right|^k\left|t_v\right|^k\right)\times$$

$$\times\left(\sum_{v=1}^{n-1}\left|\Delta_v \hat{a}_{nv}\right|\right)^{k-1}.$$

From condition (1) of Lemma 1, $\left\{\lambda_n\right\}$ is bounded, and (v);

$$I_1 = O(1)\sum_{n=1}^{m+1} n^{\delta k}\left(na_{nn}\right)^{k-1}\sum_{v=1}^{n-1}\left|\Delta_v \hat{a}_{nv}\right|\left|\lambda_v\right|^k\left|t_v\right|^k$$

$$= O(1)\sum_{n=1}^{m+1} n^{\delta k}\left(na_{nn}\right)^{k-1}\left(\sum_{v=1}^{n-1}\left|\lambda_v\right|^{k-1}\left|\lambda_v\right|\left|\Delta_v \hat{a}_{nv}\right|\left|t_v\right|^k\right)$$

$$= O(1)\sum_{v=1}^{m}\left|\lambda_v\right|\left|t_v\right|^k\sum_{n=v+1}^{m+1} n^{\delta k}\left(na_{nn}\right)^{k-1}\left|\Delta_v \hat{a}_{nv}\right|$$

$$= O(1)\sum_{v=1}^{m}\left|\lambda_v\right|\left|t_v\right|^k\sum_{n=v+1}^{m+1} n^{\delta k}\left|\Delta_v \hat{a}_{nv}\right|$$

$$= O(1)\sum_{v=1}^{m} v^{\delta k}\left|\lambda_v\right| a_{vv}\left|t_v\right|^k$$

$$= O(1)\sum_{v=1}^{m}\left|\lambda_v\right|\left[\sum_{r=1}^{v} a_{rr}\left|t_r\right|^k r^{\delta k} - \sum_{r=1}^{v-1} a_{rr}\left|t_r\right|^k r^{\delta k}\right]$$

$$= O(1)\sum_{v=1}^{m-1}\Delta\left(\left|\lambda_v\right|\right)\sum_{r=1}^{v}\left|t_r\right|^k r^{\delta k-1} + \left|\lambda_m\right|\sum_{r=1}^{m}\left|t_r\right|^k r^{\delta k-1}$$

$$= O(1)\sum_{v=1}^{m-1}\left|A_v\right| X_v + O(1)\left|\lambda_m\right| X_m$$

$$= O(1),$$

Again, using the hypothesis of the theorem and Lemma 1. Using Hölder's inequality:

$$I_2 := \sum_{n=2}^{m+1} n^{\delta k+k-1}\left|T_{n2}\right|^k = \sum_{n=2}^{m+1} n^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\left(\Delta\lambda_v\right)\frac{v+1}{v}t_v\right|^k$$

$$\leq \sum_{n=2}^{m+1} n^{\delta k+k-1}\left[\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\left|\Delta\lambda_v\right|\left|\frac{v+1}{v}\right|\left|t_v\right|\right]^k$$

$$= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\left[\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\left|\Delta\lambda_v\right|\left|t_v\right|\right]^k$$

$$= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\left[\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\left|\Delta\lambda_v\right|\left|t_v\right|^k\right]\times$$

$$\times\left[\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\left|\Delta\lambda_v\right|\right]^{k-1}.$$

From (Rhoades and Savas, 2006):

$$\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\left|\Delta\lambda_v\right| \leq Ma_{nn}.$$

Using (iii);

$$I_2 := O(1)\sum_{n=2}^{m+1} n^{\delta k}\left(na_{nn}\right)^{k-1}\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\left|\Delta\lambda_v\right|\left|t_v\right|^k$$

$$= O(1)\sum_{v=1}^{m}\left|\Delta\lambda_v\right|\left|t_v\right|^k\sum_{n=v+1}^{m+1} n^{\delta k}\left(na_{nn}\right)^{k-1}\left|\hat{a}_{n,v+1}\right|$$

$$= O(1)\sum_{v=1}^{m}\left|\Delta\lambda_v\right|\left|t_v\right|^k\sum_{n=v+1}^{m+1} n^{\delta k}\left|\hat{a}_{n,v+1}\right|.$$

Therefore;

$$I_2 := O(1)\sum_{v=1}^{m} v^{\delta k}\left|\Delta\lambda\right|\left|t_v\right|^k$$

We may write:

$$I_2 := O(1)\sum_{v=1}^{m} v^{\delta k} v\left|\Delta\lambda_v\right|\frac{\left|t_v\right|^k}{v}.$$

Using summation by parts and (vii);

$$I_2 := O(1)\sum_{v=1}^{m}\Delta\left(v\left|\Delta\lambda_v\right|\right)\sum_{r=1}^{v}\frac{1}{r}\left|t_r\right|^k r^{\delta k} + O(1)m\left|\Delta\lambda_m\right|\sum_{v=1}^{m}\frac{1}{r}\left|t_r\right|^k r^{\delta k}$$

$$= O(1)\sum_{v=1}^{m}\Delta\left(v\left|\Delta\lambda_v\right|\right)X_v + O(1)m\left|\Delta\lambda_m\right|X_m.$$

Using (vii), the properties of Lemma 1, and the fact that $\{X_m\}$ is almost increasing:

$$I_2 := O(1)\sum_{v=1}^{m} v|\Delta A_v| X_v + O(1)\sum_{v=1}^{m-1}|A_{v+1}| X_{v+1} + O(1)m|A_m| X_m$$
$$= O(1).$$

Using the hypothesis of the theorem 1, Hölder's inequality, summation by parts:

$$\sum_{n=2}^{m+1} n^{\delta k+k-1}|T_{n3}|^k = \sum_{n=2}^{m+1} n^{\delta k+k-1}\left|\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\lambda_{v+1}\frac{1}{v}t_v\right|^k$$
$$\leq \sum_{n=2}^{m+1} n^{\delta k+k-1}\left[\sum_{v=1}^{n-1}\frac{|\lambda_{v+1}|}{v}|\hat{a}_{n,v+1}||t_v|\right]^k$$
$$= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\left[\sum_{v=1}^{n-1}\frac{|\lambda_{v+1}|}{v}|t_v|^k|\hat{a}_{n,v+1}|\right]^k \times \left[\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\frac{|\lambda_{v+1}|}{v}\right]^{k-1}$$
$$= O(1)\sum_{n=2}^{m+1} n^{\delta k}(na_{nn})^{k-1}\left[\sum_{v=1}^{n-1}\frac{|\lambda_{v+1}|}{v}|t_v|^k|\hat{a}_{n,v+1}|\right] \times \left[\sum_{v=1}^{n-1}\frac{|\lambda_{v+1}|}{v}\right]^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{v=1}^{n-1}\frac{|\lambda_{v+1}|}{v}|t_v|^k|\hat{a}_{n,v+1}|$$

$$= O(1)\sum_{v=1}^{m}\frac{|\lambda_{v+1}|}{v}|t_v|^k\sum_{n=v+1}^{m+1} n^{\delta k}(na_{nn})^{k-1}|\hat{a}_{n,v+1}|$$

$$= O(1)\sum_{v=1}^{m}\frac{|\lambda_{v+1}|}{v}|t_v|^k\sum_{n=v+1}^{m+1} n^{\delta k}|\hat{a}_{n,v+1}|$$

$$= O(1)\sum_{v=1}^{m}\frac{|\lambda_{v+1}|}{v}v^{\delta k}|t_v|^k$$
$$= O(1)\sum_{v=1}^{m-1}(|\Delta\lambda_{v+1}|)X_v + O(1)|\lambda_{m+1}| X_m$$
$$= O(1)\sum_{v=1}^{m-1}(|A_{v+1}|)X_v + O(1)|\lambda_{m+1}| X_m$$
$$= O(1).$$

Finally, using (iii) and the hypothesis of the Theorem 1, we have:

$$\sum_{n=1}^{m} n^{\delta k+k-1}|T_{n4}|^k = \sum_{n=1}^{m} n^{\delta k+k-1}\left|\frac{(n+1)a_{nn}\lambda_n t_n}{n}\right|^k$$
$$= O(1)\sum_{n=1}^{m} n^{\delta k+k-1}|a_{nn}|^k|\lambda_n|^k|t_n|^k$$
$$= O(1)\sum_{n=1}^{m} n^{\delta k}(na_{nn})^{k-1}a_{nn}|\lambda_n|^{k-1}|\lambda_n||t_n|^k$$
$$= O(1)\sum_{n=1}^{m} n^{\delta k}a_{nn}|\lambda_n||t_n|^k$$
$$= O(1).$$

As in the proof of $I_1$. Setting $\delta = 0$ in the theorem yields the following Corollary.

**Corollary 1**

Let A be a triangle satisfying conditions (i) to (iii) of Theorem 1 and if $\{X_n\}$ is an almost increasing sequence such that, $|\Delta X_n| = O\left(\frac{X_n}{n}\right)$ and $\lambda_n \to 0$ as $n \to \infty$ (Savas, 2008). Suppose that there exists a sequence of numbers $\{A_n\}$ such that, it is $\delta$-quasi monotone with $\sum nX_n\delta_n < \infty$, $\sum A_n X_n$ is convergent and $|\Delta\lambda_n| \leq |A_n|$ for all n. If:

(iv) $\sum_{n=1}^{\infty}\frac{|\lambda_n|}{n} < \infty$, and

(v) $\sum_{n=1}^{\infty}\frac{1}{n}|t_n|^k = O(X_m),$ where $t_n := \frac{1}{n+1}\sum_{k=1}^{n} ka_k,$

then the series $\sum a_n\lambda_n$ is summable $|A|_k, k \geq 1.$

**Corollary 2**

Let $\{p_n\}$ be a positive sequence such that:

$P_n := \sum_{k=0}^{n} p_k \to \infty$, and satisfies:

(i) $np_n = O(P_n)$

(ii) $\sum n^{\delta k}\left|\frac{p_n}{P_n P_{n-1}}\right| = O(\frac{v^{\delta k}}{P_v})$

If $\{X_n\}$ is an almost increasing sequence such that $|\Delta X_n| = O\left(\frac{X_n}{n}\right)$ and $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers $\{A_n\}$ such that it is $\delta$-quasi monotone with $\sum nX_n\delta_n < \infty$, $\sum A_n X_n$ is convergent and $|\Delta\lambda_n| \leq |A_n|$ for all n. If;

(iii) $\sum_{n=1}^{\infty}\frac{|\lambda_n|}{n} < \infty$, and (iv) $\sum_{n=1}^{m} n^{\delta k-1}|t_n|^k = O(X_m),$

then the series $\sum a_n \lambda_n$ is summable $\left|\bar{N}, p, \delta\right|_k, k \geq 1$ *for* $0 \leq \delta < 1/k.$

***Proof***

Conditions (iii) and (iv) of Corollary 2 are, respectively, conditions (vi) and (vii) of Theorem 1. Conditions (i) and (ii) of Theorem 1 are automatically satisfied for any weighted mean method. Condition (iii) of Theorem 1 become condition (i) of Corollary 2 and conditions (iv) and (v) of Theorem 1 becomes condition (ii) of Corollary 2.

## CONCLUSION

Let $\sum a_V$ denote a series with partial sums $s_n$. For an infinite matrix A, the *nth* term of the A-transform of $\{s_n\}$ is denoted by:

$$t_n = \sum_{v=0}^{\infty} t_{nv} s_v.$$

Recently, Savas (2008), obtained an absolute summability factor theorem for lower triangular matrices. A summability factor theorem for summability $\left|A, \delta\right|_k$ as defined in (Flett, 1957) has not been studied so far. The present paper is filled up a gap in the existing literature.


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