Developing a fuzzy TOPSIS model in multiple attribute group decision making

Mohammad Anisseh and Rosnah bt Moh Yousuff*

Department of Mechanical and Manufacturing Engineering, University Putra Malaysia (UPM), 43400 UPM, Serdang, Selangor Darul Ehsan, Malaysia.

Accepted 2 February, 2011

Decision making is explained as a process through which, the solution for a problem is selected. Lack of perfect information evidently leads to uncertainty. Decision makers’ opinion are expressed through words of natural language through a process of fuzzy multi attribute group decision making problem and aggregates the performance rating with respect to all the attributes for each alternative. In this paper, a technique for order preference by similarity to ideal solution (TOPSIS) has been dealt with in the fuzzy environment based on decision makers’ viewpoint weights. The decision makers’ fuzzy decision matrixes were converted into an aggregated decision matrix to determine the most preferable choice among all possible alternatives. Finally, a numerical example was used to illustrate the procedure of the method and compare it with other extant methods.

Key words: Fuzzy TOPSIS, fuzzy numbers, group decision making, multiple attribute group decision making.

INTRODUCTION

Decision making describes the process through which, the solution of certain problems can be chosen (Hwang and Yoon, 1995). Most important decisions in organizations are made by groups of managers or experts. Managers spend much of their time in decision related meetings (Huber, 1984). Balancing tradeoffs between objectives is even more important in groups than for individuals, because conflicting objectives and opposing viewpoints are inevitably going to exist. Sycara (1991) presented a framework for problem restructuring based on the goals and goal relationships of the negotiating parties. Decision making groups can range from cooperative with very similar goals and outlooks, to antagonistic, with diametrically opposed objectives. Even in cooperative groups, conflict can arise during the decision process (Poole et al., 1991). If group members have different viewpoints, some method of aggregating preferences and reconciling differences are needed. Multi criteria decision making (MCDM) methods have been developed to solve conflicting preferences among criteria for single decision makers (Corner and Kirkwood, 1991; Korhonen et al., 1984; Saaty, 1980; Keeney and Raiffa, 1976). MCDM has proven to be an effective methodology for solving a large variety of multi criteria evaluation and ranking problems (Yeh and Chang, 2008; Hwang and Yoon, 1981). A MCDM problem can be concisely expressed in matrix format as:

\[
D = \begin{bmatrix}
C_1 & C_2 & \cdots & C_n \\
A_1 & x_{11} & x_{12} & \cdots & x_{1n} \\
A_2 & x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_m & x_{m1} & x_{m2} & \cdots & x_{mn}
\end{bmatrix}
\] (1)

\[
W = [w_1, w_2, \ldots, w_n]; \text{ where } A_1, A_2, \ldots, A_m \text{ are possible alternatives among which decision makers have to choose and } C_1, C_2, \ldots, C_n \text{ are criteria with which alternative performance are measured, } x_{ij} \text{ is the rating of alternative } A_i \text{ with respect to criterion } C_j \text{ while } w_j \text{ is the weight of criterion } C_j \text{ (Chen, 2000). Nowadays most of the problems faced by managers in their decision making and even our daily problems have different dimensions}

*Corresponding author. E-mail: rosnah@eng.upm.edu.my.
and are formulated with several variations. In other words, final decision could not be made with an optimizing variant. It is clear that solving these problems is complicated and is not possible easily; specially that most of this variants are contrary to each other and an increase in one of them can cause desirability in the other (Jung, 2001). Making decisions based on the opinions of a number of people instead of one person, causes much intricacies in terms of analysis of the decision that is not only due to access to collective agreement in ranking of alternatives, but through other circumstances like possible differences between group decisions makers who decide according to their different objectives and criteria (Fletcher, 2001).

TOPSIS is a useful technique in dealing with multi attribute or multi-criteria decision making (MADM/MCDM) problems in the real world (Hwang and Yoon, 1981). It helps decision maker(s) (DMs) organize the problems to be solved, and carry out analysis, comparisons and rankings of the alternatives. Accordingly, the selection of a suitable alternative(s) will be made (Shihet et al., 2007). The basic principle is that the chosen alternative should have the shortest distance from the positive ideal solution and the farthest distance from the negative ideal solution. In conventional MCDM methods, ratings and weights of the attributes are known precisely. Generally, DMs’ judgments are uncertain and cannot be estimated by exact numerical values. Under many conditions, crisp data are inadequate to model real-life situations; human judgments, including preferences, are often vague and preferences cannot be estimated in exact numerical values (Zhang et al., 2008).

In order to deal with vagueness of human thought, Zadeh (1965) first introduced the fuzzy set theory. A fuzzy set presents a boundary with a gradual contour, designated as membership function or membership grade (also designated as degree of compatibility or degree of truth). Let U be the universe of discourse and U = {u}, where u ∈ U. A fuzzy subset Â, of U, a real number µ ~ u (u) in the interval [0, 1] (Mario, 2000).

The basic principle is that the chosen alternative should have the shortest distance from the positive ideal solution and the farthest distance from the negative ideal solution. In conventional MCDM methods, ratings and weights of the attributes are known precisely. Generally, DMs’ judgments are uncertain and cannot be estimated by exact numerical values. Under many conditions, crisp data are inadequate to model real-life situations; human judgments, including preferences, are often vague and preferences cannot be estimated in exact numerical values (Zhang et al., 2008).

In order to deal with vagueness of human thought, Zadeh (1965) first introduced the fuzzy set theory. A fuzzy set presents a boundary with a gradual contour, designated as membership function or membership grade (also designated as degree of compatibility or degree of truth). Let U be the universe of discourse and U = {u}, where u ∈ U. A fuzzy subset Â, of U, a real number µ ~ u (u) in the interval [0, 1] (Mario, 2000).
Figure 1. The trapezoidal fuzzy membership function

Definition 2

A trapezoidal fuzzy number $M_1 = (a_1, a_2, a_3, a_4)$, $a_1 \leq a_2 \leq a_3 \leq a_4$, and its membership function is shown as (Equation 2) and (Figure 1) (Yang and Hsieh, 2008).

$$
\mu_a(x) = \begin{cases} 
0, & x \leq a_1 \\
\frac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\
1, & a_2 \leq x \leq a_3 \\
\frac{x - a_3}{a_4 - a_3}, & a_3 \leq x \leq a_4 \\
0, & x \geq a_4 
\end{cases} 
$$

Definition 3

Let $M_2$ be defined by a triplet $(b_1, b_2, b_3, b_4)$. Then, the basic operations on fuzzy trapezoidal numbers are shown as (Equation 3) (Yang and Hsieh, 2008).

$$
M_1 + M_2 = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4), \\
M_1 \times M_2 = (a_1 \times b_1, a_2 \times b_2, a_3 \times b_3, a_4 \times b_4),
$$

Table 1. Linguistic variables for the ratings are:

<table>
<thead>
<tr>
<th>Extremely poor EP</th>
<th>(0, 0, 1, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very poor VP</td>
<td>(1, 2, 3, 4)</td>
</tr>
<tr>
<td>Poor P</td>
<td>(2, 3, 4, 5)</td>
</tr>
<tr>
<td>medium poor MP</td>
<td>(3, 4, 5, 6)</td>
</tr>
<tr>
<td>Fair F</td>
<td>(4, 5, 6, 7)</td>
</tr>
<tr>
<td>Medium good MG</td>
<td>(5, 6, 7, 8)</td>
</tr>
<tr>
<td>Good G</td>
<td>(6, 7, 8, 9)</td>
</tr>
<tr>
<td>Very good VG</td>
<td>(7, 8, 9, 10)</td>
</tr>
<tr>
<td>Extremely good EG</td>
<td>(8, 9, 10, 10)</td>
</tr>
</tbody>
</table>

Table 2. Linguistic variables for the importance weight of each criterion are:

<table>
<thead>
<tr>
<th>Extremely low EL (0, 0, 0.1, 0.2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very low VL (0.1, 0.2, 0.3, 0.4)</td>
</tr>
<tr>
<td>Low L (0.2, 0.3, 0.4, 0.5)</td>
</tr>
<tr>
<td>Medium low ML (0.3, 0.4, 0.5, 0.6)</td>
</tr>
<tr>
<td>Medium M (0.4, 0.5, 0.6, 0.7)</td>
</tr>
<tr>
<td>Medium high MH (0.5, 0.6, 0.7, 0.8)</td>
</tr>
<tr>
<td>High H (0.6, 0.7, 0.8, 0.9)</td>
</tr>
<tr>
<td>Very high VH (0.7, 0.8, 0.9, 1.0)</td>
</tr>
<tr>
<td>Extremely high EH (0.8, 0.9, 1.0, 1.0)</td>
</tr>
</tbody>
</table>

TOPSIS method

Hwang and Yoon (1981) introduced the TOPSIS method based on the idea that the best alternative should have the shortest distance from an ideal solution. The procedure of TOPSIS can be expressed in a series of steps as follows: Calculate the normalized decision matrix.

$$
n_{ij} = \frac{x_{ij}}{\sqrt{\sum_{j=1}^{m} x_{ij}^2}}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n
$$

Calculate the weighted normalized decision matrix.

$$
V = N \cdot W = \begin{bmatrix}
V_{11} & \cdots & V_{1n} \\
\vdots & \ddots & \vdots \\
V_{m1} & \cdots & V_{mn}
\end{bmatrix}
$$

Where $w_i$ is the weight of the $i$th attribute or criterion, and $\sum_{j=1}^{m} w_j = 1$.

Determine the positive ideal and negative ideal solution:

$$
A^+ = \left\{ \left( \max_{i \in I} v_{ij} \right), \left( \min_{i \in J} v_{ij} \right) \right\} | i = 1, 2, \ldots, n
$$
\[
A^{-} = \left\{ \left( \min_{j} v_{ij} \mid i \in I \right), \left( \max_{j} v_{ij} \mid i \in J \right) \mid i = 1,2,\ldots,m \right\}
\]

Where \( I \) is associated with benefit criteria, and \( J \) is associated with cost criteria. Calculate the separation measures, using the \( n \)-dimensional Euclidean distance. The separation of each alternative from the ideal solution is given as:

\[
d_{i}^{+} = \left\{ \sum_{j=1}^{n} (v_{ij} - v_{ij}^{+})^{2} \right\}^{1/2}, \quad i = 1,2,\ldots,m
\]  

Similarly, the separation from the negative ideal solution is given as

\[
d_{i}^{-} = \left\{ \sum_{j=1}^{n} (v_{ij} - v_{ij}^{-})^{2} \right\}^{1/2}, \quad i = 1,2,\ldots,m
\]

Calculate the relative closeness to the ideal solution. The relative closeness of the alternative \( A_{i} \) with respect to \( A^{-} \) is defined as:

\[
cl_{i} = \frac{d_{i}^{-}}{d_{i}^{+} + d_{i}^{-}}, \quad 0 \leq cl_{i} \leq 1, \quad i = 1,2,\ldots,m
\]

Rank the preference order. For ranking alternatives using this index, we can rank alternatives in decreasing order.

THE PROPOSED METHOD

Fuzzy multiple attribute group decision making problems considered in this paper is represented as follows:

Let \( A = \{ A_{1}, A_{2}, \ldots, A_{n} \} \) be a discrete set of alternatives, \( D = \{ D_{1}, D_{2}, \ldots, D_{m} \} \) be the set of decision makers, and \( \lambda = (\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}) \) be the weight vector of decision makers, where \( \lambda_{k} \geq 0, \quad k = 1,2,\ldots,t \), and \( \sum_{k=1}^{t} \lambda_{k} = 1 \). Let \( C = \{ C_{1}, C_{2}, \ldots, C_{n} \} \) be the set of attributes, and \( w = (w_{1}, w_{2}, \ldots, w_{n}) \) be the weight vector of attributes, where \( w_{n} \geq 0, \quad n=1,2,\ldots,j \), \( \sum_{n=1}^{n} w_{n} = 1 \). The fuzzy group decision problem can be concisely expressed as a matrix format (Mahdavi et al., 2008):

\[
\overrightarrow{D}_{1} = \begin{bmatrix}
C_{1} & C_{2} & \cdots & C_{n} \\
A_{1} & \overrightarrow{x}_{11} & \overrightarrow{x}_{12} & \cdots & \overrightarrow{x}_{1n} \\
A_{2} & \overrightarrow{x}_{21} & \overrightarrow{x}_{22} & \cdots & \overrightarrow{x}_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{n} & \overrightarrow{x}_{n1} & \overrightarrow{x}_{n2} & \cdots & \overrightarrow{x}_{nn}
\end{bmatrix}
\]

\[
\overrightarrow{X} = [\overrightarrow{x}_{11}, \overrightarrow{x}_{12}, \cdots, \overrightarrow{x}_{1n}] \\
\overrightarrow{Y} = [\overrightarrow{y}_{11}, \overrightarrow{y}_{12}, \cdots, \overrightarrow{y}_{1n}]
\]

\[
\overrightarrow{W} = [\overrightarrow{w}_{1}, \overrightarrow{w}_{2}, \cdots, \overrightarrow{w}_{n}] \\
\overrightarrow{X} = [\overrightarrow{x}_{11}, \overrightarrow{x}_{12}, \cdots, \overrightarrow{x}_{1n}]
\]

Where \( \overrightarrow{X} \) and \( \overrightarrow{Y} \) are linguistic variables that can be shown by fuzzy numbers (Mahdavi et al., 2008).

First step is, identification of the evaluation criteria.
Step 2 is, generating alternatives.
Step 3 is, evaluating alternatives in terms of criteria (the values of the criterion functions are fuzzy).
Step 4 is, identifying the weight of criteria and the weight of decision makers (the values of the weights can be crisp or fuzzy).
Step 5. Construction of the fuzzy decision matrix (in fuzzy decision matrix, we suppose that, each \( x_{ij}^{k} \) is fuzzy number).

Step 6. We calculate the normalized fuzzy decision matrix as follows: For the type of revenue attributes, we have

\[
y_{ij}^{(1)} = (a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}, a_{ij}^{(4)}) = \left( \frac{a_{ij}}{\max(a_{ij})}, \frac{a_{ij}}{\max(a_{ij})}, \frac{a_{ij}}{\max(a_{ij})}, \frac{a_{ij}}{\max(a_{ij})} \right)
\]

For the type of cost attributes, we have

\[
y_{ij}^{(2)} = (a_{ij}^{(3)}, a_{ij}^{(4)}, a_{ij}^{(5)}, a_{ij}^{(6)}) = \left( \frac{\min(a_{ij})}{a_{ij}}, \frac{\min(a_{ij})}{a_{ij}}, \frac{\min(a_{ij})}{a_{ij}}, \frac{\min(a_{ij})}{a_{ij}} \right)
\]

where operator \(^{\circ}a\) is the floor operator (Li, 2002). The normalization method mentioned above is to preserve the property that the ranges of normalized fuzzy numbers belong to \([0, 1]\). In order to avoid these computations and make an easier and practical procedure all the fuzzy numbers in this interval are defined to avoid normalization method (Mahdavi et al., 2008; Wang and Lee, 2007; Saghafian and Hejazi, 2005).

Step 7. We defuzzify and aggregate DMs fuzzy decision matrices as follows for trapezoidal fuzzy numbers:

\[
g_{ij} = \frac{1}{n} \left[ \pi^{(t)}(a_{ij}^{(k)})^{(t)} \right]^{\gamma_{k}} + \left[ \pi^{(t)}(b_{ij}^{(k)})^{(t)} \right]^{\gamma_{k}} + \left[ \pi^{(t)}(c_{ij}^{(k)})^{(t)} \right]^{\gamma_{k}} + \left[ \pi^{(t)}(d_{ij}^{(k)})^{(t)} \right]^{\gamma_{k}}
\]

Where \( n=3 \) for triangular fuzzy numbers and \( n=4 \) for trapezoidal fuzzy numbers, and \( \overrightarrow{X}^{(k)} = (a_{ij}^{(k)}, b_{ij}^{(k)}, c_{ij}^{(k)}, d_{ij}^{(k)}) \) is the DMs viewpoints importance weights where \( \overrightarrow{X}^{(k)} \geq 0, \quad k = 1,2,\ldots,t \), and \( \sum_{k=1}^{t} \overrightarrow{X}^{(k)} = 1 \), so \( \sum_{k=1}^{t} \overrightarrow{X}^{(k)} = N \). Or

\[
g_{ij} = \frac{1}{n} \left[ \pi^{(t)}(a_{ij}^{(k)})^{(t)} + b_{ij}^{(k)} + c_{ij}^{(k)} + d_{ij}^{(k)})^{(t)} \right]^{\gamma_{k}}
\]

Where \( n=3 \) for triangular fuzzy numbers and \( n=4 \) for trapezoidal fuzzy numbers, and \( \overrightarrow{X}^{(k)} = (a_{ij}^{(k)}, b_{ij}^{(k)}, c_{ij}^{(k)}, d_{ij}^{(k)}) \) is the weight vector of decision makers, where \( \overrightarrow{X}^{(k)} \geq 0, \quad k = 1,2,\ldots,t \), and \( \sum_{k=1}^{t} \overrightarrow{X}^{(k)} = 1 \), so \( \sum_{k=1}^{t} \overrightarrow{X}^{(k)} = N \). Or

\[
g_{ij} = \left[ \pi^{(t)} \left( \overrightarrow{X}^{(k)} \right) \right]^{\gamma_{k}}
\]
### Table 3. The ratings of the five candidates by decision makers under all criteria.

<table>
<thead>
<tr>
<th></th>
<th>$C_1$ : Teaching</th>
<th>$C_2$ : Research</th>
<th>$C_3$ : Service</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1$</td>
<td>MP, F, MP, VG, MG</td>
<td>G, VG, MP, MG</td>
<td>G, G, VG, MG</td>
</tr>
<tr>
<td>$D_3$</td>
<td>F, F, MG, VG, MG</td>
<td>G, MG, VG, MP</td>
<td>G, F, VG, F</td>
</tr>
</tbody>
</table>

### Table 4. Normalized fuzzy performance matrix.

<table>
<thead>
<tr>
<th></th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>$A_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>(0.3, 0.44, 0.62, 0.86)</td>
<td>(0.4, 0.55, 0.75, 1)</td>
<td>(0.3, 0.44, 0.62, 0.86)</td>
<td>(0.7, 0.88, 1, 1)</td>
<td>(0.5, 0.66, 0.87, 1)</td>
</tr>
<tr>
<td>$D_1$</td>
<td>(0.6, 0.77, 1, 1)</td>
<td>(0.7, 0.88, 1, 1)</td>
<td>(0.3, 0.44, 0.62, 0.86)</td>
<td>(0.5, 0.66, 0.87, 1)</td>
<td>(0.6, 0.77, 1, 1)</td>
</tr>
<tr>
<td>$C_2$</td>
<td>(0.6, 0.77, 1, 1)</td>
<td>(0.7, 0.88, 1, 1)</td>
<td>(0.7, 0.88, 1, 1)</td>
<td>(0.3, 0.44, 0.62, 0.86)</td>
<td>(0.5, 0.66, 0.87, 1)</td>
</tr>
<tr>
<td>$D_2$</td>
<td>(0.44, 0.62, 0.86, 1)</td>
<td>(0.66, 0.87, 1, 1)</td>
<td>(0.6, 0.77, 1, 1)</td>
<td>(0.5, 0.66, 0.87, 1)</td>
<td>(0.7, 0.88, 1, 1)</td>
</tr>
<tr>
<td>$C_3$</td>
<td>(0.6, 0.77, 1, 1)</td>
<td>(0.7, 0.88, 1, 1)</td>
<td>(0.5, 0.66, 0.87, 1)</td>
<td>(0.5, 0.66, 0.87, 1)</td>
<td>(0.4, 0.55, 0.75, 1)</td>
</tr>
<tr>
<td>$D_3$</td>
<td>(0.4, 0.55, 0.75, 1)</td>
<td>(0.4, 0.55, 0.75, 1)</td>
<td>(0.5, 0.66, 0.87, 1)</td>
<td>(0.7, 0.88, 1, 1)</td>
<td>(0.5, 0.66, 0.87, 1)</td>
</tr>
</tbody>
</table>

Where $\lambda^{(k)}=(\lambda^1, \lambda^2, \ldots, \lambda^t)$ is the weight vector of decision makers, and $\lambda^{(k)} \geq 0$, $k=1,2,\ldots,t$, and $\sum_{k=1}^{t} \lambda^{(k)} = 1.0$.

So $\sum_{k=1}^{t} \lambda^{(k)} = N$. The Best Nonfuzzy Performance (BNP) value is:

$$BNP_j = \max_{i} \min_{j} \{ U_{ij} - L_j \} + \min_{i} \max_{j} \{ M_{ij} - L_j \} \forall i, j$$

Step 8. Regarding the group matrix $g_{ij}$, and the continuation of step two of TOPSIS algorithm method, alternatives were classified.

**NUMERICAL EXAMPLE**

Here, first we work out a numerical example, taken from Xu (2007), to illustrate the TOPSIS approach for decision making problems with fuzzy data. So a group decision making problem of evaluating university faculty for tenure and promotion is used. A practical use of the proposed approach involves the evaluation of university faculty for tenure and promotion. The criteria (attributes) used at some universities are $C_1$ : Teaching, $C_2$ : Research, and $C_3$ : Service (whose weight vector $w = (0.36, 0.31, 0.33)$).

Five faculty candidates (alternatives) $A = \{A_1, A_2, A_3, A_4, A_5\}$ are evaluated using the linguistic variables by three DMs $D = \{D_1, D_2, D_3\}$ whose weight vector $\lambda = (0.4, 0.5, 0.1)$ under these three attributes, as listed in (Table 3).

Linguistic evaluations (shown in Table 3) are converted into trapezoidal fuzzy numbers to construct a fuzzy decision matrix. Constructed normalized fuzzy decision matrix by (Equation 11) is shown in (Table 4).

To aggregate all the decision matrixes $D = \{D_1, D_2, D_3\}$ into a collective decision matrix we utilized (Equation 12) as it could also be used with (Equation 13) or (Equation 14). (Table 5) shows group matrix aggregation based on (Equation 12).

For example, the first element of the matrix $g_{ij}$ based on (Equation 12) was calculated as follows:

$$g_{ij} = \frac{1}{4} \times \left[ 0.34^{11} \times 0.44^{15} \times 0.62^{34} + 0.44^{14} \times 0.62^{25} \times 0.55^{31} \right] = 0.64$$

Regarding the group matrix $g_{ij}$, and the continuation of
step two of TOPSIS algorithm method, alternatives were classified. Therefore, the ranking order of five university faculty will be as follows: $A_2 < A_1 < A_3 < A_4 < A_5$. So, university faculty $A_2$ is the best faculty among the five faculties, and $A_1$ is the worst faculty.

The proposed method is compared with other methods, with (Table 6) listing the results of the comparison. As it could be observed in the following table, the best selection made by the proposed method is comparable with the five already established methods which is expressive in itself and approves of the reliability and validity of the proposed method.

### Conclusion

In group multi criteria decision making problem with linguistic variables, the DMs may have vague information, limited attention and information processing capabilities. Therefore, in this paper TOPSIS for fuzzy environment has been extended based on decision makers’ viewpoint weights. We converted the DMs fuzzy decision matrix into an aggregated decision matrix to determine the most preferable choice among all possible alternatives. The method accounts for both homogeneous and heterogeneous group decision making. Moreover, the ease of its application seems to make it more practical and applicable compared to the other methods.

### REFERENCES


