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A different approach to the set of fuzzy numbers

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In this paper, we showed that every element of the set F of all fuzzy numbers can be abstractly considered as an element of the nonseparable and nonreflexive Banach algebra ϑ. Banach lattice structure of ϑ is also investigated according to a partial order we defined.

Key words: Fuzzy numbers, Banach space.

INTRODUCTION

In order to study the control problems of complicate systems and dealing with fuzzy information, Zadeh (1965) introduced fuzzy set theory, describing fuzziness mathematically for the first time. Then the study of mathematics began to explore fuzziness, subsequently, several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka (1986) introduced bounded and covergent sequences of fuzzy numbers and studied their some properties. Matloka (1986) also has shown that every convergent sequence of fuzzy numbers is bounded. Later on sequences of fuzzy numbers have been discussed in the studies by Nanda (1989), Nuray (1998), Kwon (2000), Savaş (2000), Wu (2002), Bilgin (2003), Aytar and Pehlivan (2006) and Tuncer and Benli (2007).

Just one application amongst many other concepts springing up as results of fuzziness, Dubois and Prade (1978) introduced the notion of fuzzy numbers (fuzzy sets with certain properties) and defined the basic operations of addition, subtraction, multiplication, and division. A slightly modified definition of fuzzy numbers was presented by Goetschel and Voxman (1983). Some other less restrictive definitions of fuzzy numbers can be found in literature and there has been many studies on fuzzy numbers in different views such as their linear and topological structures. We also refer the readers to some recent studies concerning diference sequence spaces of fuzzy numbers (Altin et al., 2007; Altinok et al, 2009; Çolak et al., 2009) and statistical summability (C,1) for sequences of fuzzy real numbers (Altin et al., 2010).

In the present paper, we established a normed space structure on the set F of all fuzzy numbers. Using the customary vector space operations of Goetschel and Voxman (1986), we constructed a nonseparable and nonreflexive unital Banach algebra ϑ. That is, we impose a norm on ϑ that satisfies multiplicative inequality for the third operation which makes ϑ a Banach algebra. We reduce the norm of ϑ on F. As a result, we showed F can be considered to have a normed algebra structure. Furthermore, in this study, it has been shown that the space ϑ is nonreflexive and it has Banach lattice structure according to a partial order as defined on the space ϑ. Although, Benli and Soybaş (2010) studied the set F of fuzzy numbers in a way similar to the present study, Banach lattice structure with a different norm is first regarded in this study.

MATERIALS AND METHODS

We now give the following definitions which will be needed in the sequel. For the sake of both completeness and easy comprehension we will give in the main results section of our study, the following preparation which was first introduced by Goetschel

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and Voxman (1986) as follows;

Let F be the set of all fuzzy numbers, that is, F is the set of all functions $u : IR \to [0,1]$ with the following properties:

1. u is normal, i.e. there is a unique element $x_0 \in IR$ such that $u(x_0) = 1$;
2. u is fuzzy convex, i.e. for any $\lambda \in 0,1]$ and for any $x, y \in IR$,

$$u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u\{y)\} \quad (1)$$

3. u is upper semicontinuous;
4. the support of u, $[u]^0 = \text{closure}\{x \in IR: u(x) > 0\}$, is compact.

We start by recalling the linear structure of the set F of all fuzzy numbers as follows. For $u \in F$ and $0 < \alpha \leq 1$, the α *-level set* of u is $[u]^\alpha = \{x \in IR : u(x) \geq \alpha\}$. It is well known (Dubois and Prade, 1980) that the α -level sets of a fuzzy number are closed intervals or singletons, with $[u]^\alpha \subset [u]^\beta$ if $\alpha < \beta$. If $[u]^\alpha$ is an interval, we will write $[u]^\alpha = [[u]^\alpha_-, [u]^\alpha_+]$. If $[u]^\alpha$ is a singleton, we will use the same notation, with $[u]^\alpha_- = [u]^\alpha_+$. For any two fuzzy numbers u and v, the extension principle of Dubois and Prade (1980) is used to define the sum $u + v$. It can be shown that this definition yields a fuzzy number whose α -level sets are:

$$[u+v]^\alpha = [[u]^\alpha_- + [v]^\alpha_-, [u]^\alpha_+ + [v]^\alpha_+] \quad (2)$$

Similarly, if $u \in F$ and $a \in IR$ the extension principle yields:

$$[au]^\alpha = \begin{cases} [a[u]^\alpha_-, a[u]^\alpha_+], \text{if } a \geq 0 \\ [a[u]^\alpha_+, a[u]^\alpha_-], \text{otherwise} \end{cases} \quad (3)$$

However, if u and v are two fuzzy numbers, the product $u \otimes v$ is a little more complicated. By the extension principle:

$$[u \otimes v]^\alpha = \{xy : x \in [u]^\alpha \text{ and } y \in [v]^\alpha\} \quad (4)$$

Thus, $[u \otimes v]^\alpha$ is an interval of the form $[a, b]$, where,

$$a = \min\{[u]^\alpha_-[v]^\alpha_-, [u]^\alpha_-[v]^\alpha_+, [u]^\alpha_+[v]^\alpha_-, [u]^\alpha_+[v]^\alpha_+\} \quad (5)$$

and

$$b = \max\{[u]^\alpha_-[v]^\alpha_-, [u]^\alpha_-[v]^\alpha_+, [u]^\alpha_+[v]^\alpha_-, [u]^\alpha_+[v]^\alpha_+\} \quad (6)$$

Suppose that $u : IR \to [0,1]$ is a fuzzy set and for $0 \leq r \leq 1$, define $C_r(u)$ by:

$$C_r(u) = \begin{cases} \{(x,r): u(x) \geq r\}, \text{ if } 0 < r \leq 1 \\ \text{cl(supp u)}, \quad \text{if } r = 0 \end{cases}$$

Where cl(supp u) denotes the closure of the support of u. Then it is easily established that u is a fuzzy number if and only if:

i) $C_r(u)$ is a closed and bounded interval for each r, $0 \leq r \leq 1$,

ii) $C_1(u) \neq \emptyset$.

From this characterization of fuzzy numbers it can be seen from the study of Goetschel and Voxman (1986), that a fuzzy number is determined by the endpoints of the intervals C_r. Thus a fuzzy number u can be identified with the parameterized couples:

$$\{(a(r), b(r)): 0 \leq r \leq 1\} \quad (7)$$

Where $a(r)$ denotes the left hand endpoint of $C_r(u)$ and $b(r)$ denotes the right hand endpoint. This leads to the following characterization of a fuzzy number in terms of the two 'endpoint' functions a and b. Following theorem is due to R. Goetschel and W. Voxman.

Theorem 2.1. (Goetschel and Voxman,1986) Suppose that $a : [0,1] \to IR$ and $b : [0,1] \to IR$ satisfy the conditions:

i) a is a bounded increasing function,

ii) b is a bounded decreasing function,

iii) $a(1) \leq b(1)$,

iv) for $0 < k \leq 1$, $\lim_{r \to k^-} a(r) = a(k)$ and $\lim_{r \to k^-} b(r) = b(k)$,

v) $\lim_{r \to 0^+} a(r) = a(0)$ and $\lim_{r \to 0^+} b(r) = b(0)$. Then $u : IR \to [0,1]$ defined by:

$$u(x) = \sup\{r : a(r) \leq x \leq b(r)\} \quad (8)$$

Is a fuzzy number with parameterization given by (7). Moreover, if $u : IR \to [0,1]$ is a fuzzy number with parameterization given by (7), then the functions a and b satisfy conditions (i)-(v). Suppose that $u, v \in F$ are fuzzy numbers represented by $\{(a(r), b(r)): 0 \leq r \leq 1\}$ and $\{(c(r), d(r)): 0 \leq r \leq 1\}$, respectively. Then it is easily verified that the Dubois-Prade (1978) definition of the addition of fuzzy numbers:

$$(u + v)(z) = \max_{x+y=z} \min(u(x), v(y)) \quad (9)$$

Is equivalent to the 'vector' addition of the parametric representations of u and v:

$$\{(\mathrm{a(r), b(r)}):0 \le \mathrm{r} \le 1\} + \{(\mathrm{c(r), d(r)}):0 \le \mathrm{r} \le 1\}$$
$$= \{(a(r)+\mathrm{c(r), b(r)+d(r)}):0 \le \mathrm{r} \le 1\} \quad (10)$$

In the sense that the parameterized couples on the right hand side of (10) define a fuzzy number (as in (8)) that is equal to $u+v$ as defined by (9). To deal with subtraction Dubois and Prade define the 'opposite' of a fuzzy number u to be the fuzzy number v where $\mathrm{v(x) = u(-x)}$.

In parametric form, if u is represented by $\{(\mathrm{a(r), b(r)}):0 \le \mathrm{r} \le 1\}$ then v corresponds to the fuzzy number:

$$\{(-b(\mathrm{r}), -\mathrm{a(r)}):0 \le \mathrm{r} \le 1\} \quad (11)$$

Rather than take this approach it can be used the vector 'opposite' of u:

$$\{(-a(\mathrm{r}), -\mathrm{b(r)}):0 \le \mathrm{r} \le 1\} \quad (12)$$

To represent $-u$. It should be noted that $-u$ is not a fuzzy number.

For each fuzzy number u parameterized by:

$$\{(\mathrm{a(r), b(r)}):0 \le \mathrm{r} \le 1\}$$

and each real number c, let the 'scalar' product cu be defined by:

$$\mathrm{cu} = \{(\mathrm{ca(r), cb(r)}):0 \le \mathrm{r} \le 1\}. \quad (13)$$

With this definition it is clear that if u is a fuzzy number, then:

$$-u = (-1)u.$$

The family of parametric representations of members of F and the parametric representations of their 'opposites' form subsets of the vector space:

$$\vartheta = \{(\mathrm{a(r), b(r)}):0 \le \mathrm{r} \le 1, a:[0,1] \to IR \text{ and } b:[0,1] \to IR \text{ are bounded functions}\}$$

Where addition and scalar multiplication in ϑ are defined by (10) and (13). In addition, we define a product $\otimes$, on ϑ as a third operation, such that:

$$x_1 \otimes x_2 = (a_1(r)a_2(r), b_1(r)b_2(r)) \quad (14)$$

Where $x_1 = (a_1(r), b_1(r))$ and $x_2 = (a_2(r), b_2(r))$. We need this third operation in order to investigate algebraic structure of ϑ.

Taking the parametric representations into account, the set F of all fuzzy numbers can be considered as a subset of the vector space ϑ in respect to the same operations of addition and scalar multiplication.

RESULTS

We defined a norm on ϑ in the next lemma.

Lemma 3.1: Let $x=(a(r),b(r))$ be an element of the vector space ϑ. Then $\|x\| = \sup_{0\le r\le 1} \max\{|a(r)|,|b(r)|\}$ defines a norm on ϑ.

Proof. Let $x=(a_1(r),b_1(r))$ and $y=(a_2(r),b_2(r))$ be given as two elements of the vector space ϑ.

i) $\|x\| = \sup_{0\le r\le 1} \max\{|a_1(r)|,|b_1(r)|\} = 0 \Leftrightarrow (a_1(r),b_1(r)) = 0 \Leftrightarrow x = 0$

ii) For each $t \in IR$ we have $\|tx\| = |t|.\|x\|$

iii) $\|x+y\| = \sup_{0\le r\le 1} \max\{|a_1(r)+a_2(r)|,|b_1(r)+b_2(r)|\}$
$\le \sup_{0\le r\le 1} \max\{|a_1(r)|+|a_2(r)|,|b_1(r)|+|b_2(r)|\}$
$\le \sup_{0\le r\le 1} \max\{|a_1(r)|,|b_1(r)|\} + \sup_{0\le r\le 1} \max\{|a_2(r)|,|b_2(r)|\}$
$\le \|x\| + \|y\|$

F can be considered as a subset of ϑ in respect to an addition and a scalar multiplication mentioned in the preliminaries part. If we reduce this norm from ϑ on F, we have the norm of any element u as:

$$\|u\| = \sup_{0\le r\le 1} \max\{|[u]_-^\alpha|,|[u]_+^\alpha|\} \quad (11)$$

Following corollary can be given as an immediate consequence of Hahn-Banach extention theorem.

Corollary 3.2: Let $f:F \to IR$ be a linear functional and $f(x) \le \|x\|$ on F. Then there exist a linear functional $T:\vartheta \to IR$ such that $T(x) = f(x), (x \in F)$ and $\|T\|_{op} \le 1$, where $\|T\|_{op}$ means the operator norm of T.

Lemma 3.3. Let A and B be two bounded sets of nonnegative real numbers. Then the equality $\sup(A.B) = \sup A.\sup B$ is satisfied, where $A.B = \{ab : a \in A, b \in B\}$.

Theorem 3.4. The product $\otimes$ on ϑ satisfies the multiplication inequality:

$$\|x_1 \otimes x_2\| \leq \|x_1\| \cdot \|x_2\|$$

Whenever $x_1, x_2 \in \vartheta$.

Proof. For an arbitrary $0 \leq r \leq 1$ it is clear that:

$$\max\{|a_1(r)a_2(r)|, |b_1(r)b_2(r)|\} = \max\{|a_1(r)||a_2(r)|, |b_1(r)||b_2(r)|\}.$$

Suppose that:

$$|a_1(r)||a_2(r)| \leq |b_1(r)||b_2(r)|.$$

Then we have:

$$\max\{|a_1(r)||a_2(r)|, |b_1(r)||b_2(r)|\} = |b_1(r)||b_2(r)|$$

$$\leq \max\{|a_1(r)|, |b_1(r)|\}.\max\{|a_2(r)|, |b_2(r)|\}$$

By taking supremums of each sides of the inequality and considering Lemma 3.3 we have:

$$\|x_1 \otimes x_2\| = \|(a_1(r)a_2(r), b_1(r)b_2(r))\|$$
$$= \sup_{0 \leq r \leq 1} \max\{|a_1(r)a_2(r)|, |b_1(r)b_2(r)|\}$$
$$= \sup_{0 \leq r \leq 1} \max\{|a_1(r)||a_2(r)|, |b_1(r)||b_2(r)|\}$$

$$\leq \sup_{0 \leq r \leq 1} \max\{|a_1(r)|, |b_1(r)|\}.\sup_{0 \leq r \leq 1} \max\{|a_2(r)|, |b_2(r)|\}$$

$$\leq \|x_1\|.\|x_2\|$$

Analogously, if $|a_1(r)||a_2(r)| \geq |b_1(r)||b_2(r)|$ we have the same inequality.

Theorem 3.5. ϑ is a Banach space.

Proof. Let (x_n) be a Cauchy sequence in the normed space ϑ, where:

$$(x_n) = ((a_1(r), b_1(r)), (a_2(r), b_2(r)), \ldots.., (a_n(r), b_n(r)), \ldots.).$$

Then for every $\varepsilon \rangle 0$, there exists a natural number k such that for every $n, m \geq k$ the inequality $\|x_n - x_m\| \langle \varepsilon$ is satisfied. Then for every $n, m \geq k$ we have:

$$\|(a_n(r), b_n(r)) - (a_m(r), b_m(r))\| \langle \varepsilon$$

By definition of the norm:

$$\sup_{0 \leq r \leq 1} \max\{|a_n(r) - a_m(r)|, |b_n(r) - b_m(r)|\} \langle \varepsilon$$

Hence we have $\sup_{0 \leq r \leq 1}\{|a_n(r) - a_m(r)|\} \langle \varepsilon$ and $\sup_{0 \leq r \leq 1}\{|b_n(r) - b_m(r)|\} \langle \varepsilon$.

Then for every $r \in [0,1]$ we have $|a_n(r) - a_m(r)| \langle \varepsilon$ and $|b_n(r) - b_m(r)| \langle \varepsilon$, therefore $(a_n(r))$ and $(b_n(r))$ are two Cauchy sequences. Since R is complete, $(a_n(r))$ and $(b_n(r))$ become two convergent sequences in R. Then we have two real numbers a_r and b_r such that $(a_n(r))$ and $(b_n(r))$ converge to a_r and b_r respectively.

Now we define $a : I \to IR$ and $b : I \to IR$ by $a(r) = a_r$ and $b(r) = b_r$ for each $r \in I$.
Let an element x be defined by $x = (a(r), b(r))$. It is clear that $x \in \vartheta$. For each $n, m \geq k$:

$$\|x_n - x\| = \sup_{0 \leq r \leq 1} \max\{|a_n(r) - a_r|, |b_n(r) - b_r|\}$$

$$\leq \sup_{0 \leq r \leq 1} \max\{\varepsilon, \varepsilon\} = \varepsilon.$$

Hence, the sequence x_n converges to x which means ϑ is a Banach space.

Theorem 3.6. $(\vartheta, \|\cdot\|)$ is not separable.

Proof. Suppose that B is the collection of all fuzzy functions with domain $[0,1]$ that are indicator functions of intervals of the form $[0, s]$ where $0 \leq s \leq 1$. Take $A = B \times B$. Then A is an uncountable subset of ϑ such that:

$$\|x-y\|=\sup_{0\le r\le 1}\max\{|a_1(r)-a_2(r)|,|b_1(r)-b_2(r)|\}$$
$$=1$$

Whenever $x=(a_1(r),b_1(r))$ and $y=(a_2(r),b_2(r))$ are different members of A. Hence, B is not separable. It is well known that every subset of an separable set is also separable. Therefore, ϑ is not separable.

Theorem 3.7. ϑ is a unital Banach algebra.

Proof. As mentioned in the previous part, $(\vartheta,+,\cdot)$ is a vector space. For all x_1,x_2,x_3 in ϑ and every scalar α, it is clear that:

i) $x_1\otimes(x_2\otimes x_3)=(x_1\otimes x_2)\otimes x_3$

ii) $x_1\otimes(x_2+x_3)=(x_1\otimes x_2)+(x_1\otimes x_3)$ and $(x_1+x_2)\otimes x_3=(x_1\otimes x_3)+(x_2\otimes x_3)$

iii) $\alpha\cdot(x_1\otimes x_2)=(\alpha\cdot x_1)\otimes x_2 = x_1\otimes(\alpha\cdot x_2)$.

Then $(\vartheta,+,\cdot,\otimes)$ is an algebra. This algebra clearly becomes unital with its identity $I=(1,1)$. With our norm $\|\cdot\|,\vartheta$ is Banach space by Theorem 3.5. By Theorem 3.4 , ϑ becomes a Banach algebra.

Note that the product by (4) on F and the product by (14) on ϑ coincides on F.

A partial order on ϑ can be given by declaring that $(a_1(r),b_1(r))\le(a_2(r),b_2(r))$ when $a_1(r)\le a_2(r)$ and $b_1(r)\le b_2(r)$.

Theorem 3.8. ϑ is a Banach lattice.

Proof. If $x=(a_1(r),b_1(r)), y=(a_2(r),b_2(r))\in\vartheta$ and $x\le y$ then:

i) $(a_1(r),b_1(r))+(a_3(r),b_3(r))\le(a_2(r),b_2(r))+(a_3(r),b_3(r))$ whenever $z=(a_3(r),b_3(r))$

ii) $tx\le ty$ whenever $t>0$.

Hence ϑ is an ordered vector space over IR. Since every pair of elements of ϑ has a least upper bound ϑ is a vector lattice. Let the absolute value of x be defined by the formula $|x|=x\vee(-x)$. Since $\|x\|\le\|y\|$ whenever $x,y\in\vartheta$ and $|x|\le|x|$, ϑ is a normed lattice. Since ϑ is a Banach space with respect to our norm by Theorem 3.5, ϑ is a Banach lattice.

Note that if we reduce the partial order, mentioned above, onto the set F of all fuzzy numbers, we have, for $u,v\in F$:

$u\le v$ iff $[u]_-^\alpha\le[\mathrm{v}]_-^\alpha$ and $[\mathrm{u}]_+^\alpha\le[\mathrm{v}]_+^\alpha$ for all $\alpha\in[0,1]$.

Theorem 3.9. ϑ is not reflexive.

Proof. Let A be the space defined as the cartesian product $A=l_\infty\times l_\infty$ Since l_∞ is not reflexive, A is not reflexive. Let $u\in A$. Then, there exists $u=(\alpha,\beta)$ such that $\alpha=(x_n),\beta=(y_n)\in l_\infty$. Lets define:

$$a_1(r)=\begin{cases}x_n;\ r=\frac{1}{n}0;otherwise\end{cases}$$

$$b_1(r)=\begin{cases}y_n;\ r=\frac{1}{n}0;otherwise\end{cases}$$

It is clear that $x=(a_1(r),b_1(r))\in\vartheta$. Hence, ϑ contains a closed subspace which is a isomorphic copy of A. Since A is not reflexive, ϑ is not reflexive.

DISCUSSION

As far as we are concerned, the set F of all fuzzy numbers itself has not been investigated in respect to the normed space construction apart from Benli and Soybaş (2010). One must not confuse the concept of fuzzy norm (recall fuzzy normed spaces) with the concept of classical norm which we define on the set F of all fuzzy numbers itself. In this paper, we introduced a norm on F satisfying the multiplicative inequality. Hence, we saw that the set F of all fuzzy numbers becomes a normed algebra, which is beyond our expectation of being just a normed space at the beginning of the study. We also showed that ϑ is nonreflexive and it has Banach lattice structure. However, we do not know whether F is closed in ϑ. That is, we do not know whether the limit of any convergent sequence of fuzzy numbers in respect to the norm we defined in Lemma 3.1 has to be a fuzzy number. If this was true, F would be a Banach subalgebra of ϑ in abstract mean. We expect that our result may provide a useful example of normed algebra for analysists to solve various problems in the field. For instance, we do not know what the dual space of ϑ with respect to the norm. If the dual space of ϑ could be determined, there would be more points concerning the normed space contruction of ϑ to be focused. For example, the concept of weak topology on the space ϑ could be investigated, which may be connected to geometric properties of the space ϑ (Grothendieck property or Dunford- Pettis property and many other properties, see Diestel's book (1993)). It is well-known that most geometric properties of Banach spaces have been related to the concept of weak convergence of certain bounded linear operators Freedman and Ülger, 2000; Soybaş and Argün, 2006; Ülger, 2001). From the point of view, for instance, we can ask the question: Is every bounded linear operator from the space ϑ to its dual weakly compact? To be able to answer this question can be a focus for a different study.

REFERENCES

Altin Y, Et M, Basarir M (2007). On some generalized difference sequences of fuzzy numbers. Kuwait J. Sci. Eng., 34(1A): 1-14.

Altin Y, Mursaleen M, Altinok H (2010). Statistical summability (C,1) for sequences of fuzzy real numbers and a Tauberian theorem. J. Intell. Fuzzy Syst., 21: 3790-384.

Altinok H, Çolak R, Et M (2009). λ-Difference sequence spaces of fuzzy Numbers. Fuzzy Sets Syst., 160(21): 3128-3139.

Aytar S, Pehlivan S (2006). Statistically monotonic and statistically bounded sequences of fuzzy numbers. Inf. Sci., 176 734-744.

Benli FB, Soybaş, D (2010). On Embedding Fuzzy Numbers into a Banach Space. J. Math. Sci. Adv. Appl., 5(1) 103-112.

Bilgin T (2003). Δ -statistical and strong Δ -Cesaro convergence of sequences of fuzzy numbers. Math. Commun., 8: 95-100.

Çolak R, Altinok H, Et M (2009). Generalized difference sequences of numbers. Chaos, Solitons Fractals, 40: 1106-1117.

Diestel J (1993). Sequences and Series in Banach spaces. Springer-Verlag. Newyork.

Dubois D, Prade H (1978). Operations on fuzzy numbers. Internat. J. Syst., Sci., 9: 613-626.

Dubois D, Prade H (1980). Fuzzy Sets and Systems: Theory and Applications. Academic Pres. New York.

Freedman W, Ülger A (2000). The Phillips Properties. Amer. Math. Soc. Proc., 128: 2137-2145.

Goetschel R, Voxman W (1983), Topological properties of fuzzy numbers. Fuzzy Sets and Systems. 9: 87-99.

Goetschel R, Voxman W (1986), Elemantary Fuzzy Calculus, Fuzzy Sets Syst., 18: 31-43.

Kwon JS (2000). On statistical and p-Cesaro convergence of fuzzy numbers. Korean J. Comput. Appl. Math., 7(1): 95-203.

Matloka M (1986). Sequences of fuzzy numbers. Busefal, 28: 28-37.

Nanda S (1989). On sequences of fuzzy numbers. Fuzzy Sets Syst., 33: 123-126.

Nuray F (1998). Lacunary statistical convergence of sequences of fuzzy numbers. Fuzzy Sets Syst., 99: 353-356.

Savas E (2000), On strongly λ-summable sequences of fuzzy numbers. Inform. Sci., 125: 181-186.

Soybaş D, Argün Z (2006). On the relationship between the weak Phillips property and Arens regularity. Indian J. Math., 48(2): 139-152

Tuncer N, Benli FB (2007). λ-Statistical limit points of the sequences of fuzzy numbers. Inform. Sci.. 177: 3297-3304.

Ülger A (2001). The weak Phillips property, Colloquium Math., 87: 147-158.

Wu C, Wang G (2002). Convergence of sequences of fuzzy numbers and fixed point theorems for increasing fuzzy mappings and application. Fuzzy Sets Syst., 130: 383-390.

Zadeh LA (1965). Fuzzy Sets. Inf. Control, 8: 338-353.