academicJournals

Vol. 9(11), pp. 471-482, 15 June, 2014
DOI: 10.5897/SRE2013.5769
Article Number: E6DB6B745213
ISSN 1992-2248 © 2014

http://www.academicjournals.org/SRE

Scientific Research and Essays

Numerical solutions for the nonlinear partial fractional Zakharov-Kuznetsov equations with time and space fractional

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Received 8 December, 2013; Accepted 7 May, 2014

In this article, we implement relatively analytical techniques such as the homotopy perturbation method and homotopy analysis method to solve nonlinear partial fractional differential Zakharov-Kuznetsov equations. The fractional derivatives are described in the Caputo sense. We compare between the approximate solutions obtained by the homotopy perturbation method and the approximate solutions obtained by homotopy analysis method. Also we make the figures compare between the approximate solutions. We compare between the approximate solutions and the exact solutions for the partial fractional differential equations when $\alpha, \beta, \gamma \to 1$.

Key words: Zakharov-Kuznetsov equations, the fractional derivatives, the homotopy perturbation method, the homotopy perturbation method, the approximate solutions.

INTRODUCTION

In recent years, fractional differential equations have gained much attention as they are widely used to describe various complex phenomena in many fields such as the fluid flow, signal processing, control theory, systems identification, biology and other areas. Several fields of application of fractional differentiation and fractional integration are already well established, some others have just started. Many applications of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems and astrophysics (Kilbas et al., 2006; Podlubny, 1999; Samko et al., 1993; El-Sayed, 1996; Herzallah et al., 2010, 2011; Magin, 2006; West et al., 2003; Jesus and Machado, 2008; Agrawal and Baleanu, 2007; Tarasov, 2008). Numerical and analytical methods have included the Adomian decomposition method (ADM) (Daftardar-Gejji and Bhalekar, 2008; Herzallah and Gepreel, 2012), the variational iteration method (VIM) (Sweilam et al., 2007), the homotopy perturbation method (Golbabai and Sayevand, 2010), and homotopy analysis method (Gepreel and Mohamed, 2013).

Consider the Zakharov-Kuznetsov ZK (m,n,k) equation:

$$u_t + a(u^m)_x + b(u^n)_{xxx} + c(u^k)_{yyx} = 0, \quad m,n,k \neq 0, \quad (1)$$

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where a,b,c are arbitrary constants and m,n,k are integers. This equation governs the behavior of weakly nonlinear ion-acoustic waves in plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field (Monro and Parkers, 1999). The Zakharov–Kuznetsov equation supports stable lump solitary waves. This makes the Zakharov–Kuznetsov equation a very attractive model equation for use in the study of vortices in geophysical flows (Molliq and Batiha, 2012; Hammouch and Mekkaoui, 2013; Golbabai and Sayevanda, 2012).

Biazar et al. (2009) applied the homotopy perturbation method to solve the Zakharov-Kuznetsov ZK (m, n, k) equations. Hesam et al. (2012) studied (1) while applying the differential transform method to obtain its approximate solutions.

In this paper, we give a new model of the nonlinear fractional Zakharov-Kuznetsov ZK (2,2,2) equation in the following form:

$$D_t^{\alpha}u + a D_x^{\beta}(u^2) + b D_x^{3\beta}(u^2) + c D_x^{\beta} D_y^{2\gamma}(u^2) = 0, \quad t > 0, \quad 0 < \alpha, \beta, \gamma \le 1, \tag{2}$$

where $D_t^{\alpha}, D_x^{\beta}, D_y^{\gamma}$ denotes the fractional derivative of order α, β, γ with respect to t, x, y respectively. We will implement HPM and HAM to obtain approximate solutions of the nonlinear fractional Zakharov-Kuznetsov ZK (2,2,2) equation.

PRELIMINARIES AND NOTATION

Here, we give some basic definitions and properties of the fractional calculus theory which will be used further in this work. Podlubny (1999) revealed further details on this. For the finite derivative in [a, b], we define the following fractional integral and derivatives.

Definition 1. A real function $f(x), x > 0,$ is said to be in the space C_{μ}, $\mu \in R,$ if there exists a real number $(P > \mu)$ such that $f(x) = x^{P} f_1(x),$ where $f_1(x) \in C(0, \infty),$ and it is said to be in the space C_{μ}^{m} if $f^{m} \in C_{\mu}, m \in N$.

Definition 2. The Riemann–Liouville fractional integral operator of order $\alpha \ge 0$ of a function $f \in C_{\mu}, \mu \ge -1,$ is defined as

$$J_t^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u)\, du, \ \alpha > 0, t > 0, J^0 f(x) = f(x). \tag{3}$$

Properties of the operator J^{α} can be found in Podlubny (1999); we mention only the following:

For $f \in C_{\mu}, \mu \ge -1, \quad \alpha, \beta \ge 0,$ and $\gamma > -1:$

(a) $J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x),$

(b) $J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x),$

(c) $J^{\alpha} x^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$ (4)

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^{α} proposed by Caputo in his work on the theory of viscoelasticity (Podlubny, 1999).

Definition 3. For $\alpha > 0$ the Caputo fractional derivative of order α on the whole space, denoted by ${}^{C}D_{+}^{\alpha}$, is defined by

$${}^{C}D_{+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{x} (x-\xi)^{n-\alpha-1} D^{n} f(\xi)\, d(\xi). \tag{5}$$

THE HOMOTOPY PERTURBATION METHOD

To illustrate the basic idea of this method (Golbabai and Sayevand, 2011), we consider the following nonlinear fractional differential equation:

$$D_t^{\alpha} u(\bar{x},t) = f(\bar{x},t) - Lu(\bar{x},t) - Nu(\bar{x},t), \quad m-1 < \alpha < m, \quad m \in N, \ t \ge 0, \bar{x} \in R^{n}, \tag{6}$$

subject to the initial and boundary conditions

$$u^{(i)}(\bar{0},0) = c_i, \qquad B\left(u, \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial t}\right) = 0, \qquad i = 0,1,...,m-1, \qquad j = 1,2,...,n, \tag{7}$$

where L is a linear operator, while N is a nonlinear operator, f is a known analytical function and D_t^{α} denotes the fractional derivative in the Caputo sense. The solution u is assumed to be a causal function of time, that is, vanishing for $t < 0.$ Also $u^{(i)}(\bar{x},t)$ is the i^{th} derivative of u , $c_i = 0,1,...,m-1$ are the specified initial conditions and B is a boundary operator.

Applying He (2006) homotopy perturbation technique, we can construct the following simple homotopy

$$(1-p) D_t^{\alpha} u(\bar{x},t) + p\, [\, D_t^{\alpha} u(\bar{x},t) + Lu(\bar{x},t) + Nu(\bar{x},t) - f(\bar{x},t)] = 0, \quad p \in [0,1], \tag{8}$$

or

$$D_t^{\alpha} u(\bar{x},t) + p\ [\ Lu(\bar{x},t) + Nu(\bar{x},t) - f\ (\bar{x},t)\ ] = 0, \qquad p \in [0,1]. \tag{9}$$

The homotopy parameter p always changes from zero to unity. In the case $p = 0,$ Equation (8) or (9) becomes

$$D_t^{\alpha} u(\bar{x},t) = 0, \tag{10}$$

and when $p = 1,$ Equation (8) or (9) turns out to be the original fractional differential equation. Applying the homotopy perturbation method, we use the homotopy parameter p to expand the solution into the following form

$$u(\bar{x},t) = u_0(\bar{x},t) + p\ u_1(\bar{x},t) + p^2\ u_2(\bar{x},t) + p^3\ u_3(\bar{x},t) + ... \tag{11}$$

For nonlinear problems, let us set $Nu(\bar{x},t) = S(\bar{x},t)$. Substituting Equation (11) into (8) or (9) and equating the terms with identical powers of p , we can obtain a series of equations of the form

$$\begin{aligned}
&p^0 : D_t^{\alpha} u(\bar{x},t) = 0,\\
&p^1 : D_t^{\alpha} u_1(\bar{x},t) = -Lu_0(\bar{x},t) - S_0(u_0(\bar{x},t)) + f\ (\bar{x},t)\ ],\\
&p^2 : D_t^{\alpha} u_2(\bar{x},t) = -Lu_1(\bar{x},t) - S_1(u_0(\bar{x},t), u_1(\bar{x},t)),\\
&p^3 : D_t^{\alpha} u_3(\bar{x},t) = -Lu_2(\bar{x},t) - S_2(u_0(\bar{x},t), u_1(\bar{x},t), u_2(\bar{x},t)),\\
&\vdots
\end{aligned} \tag{12}$$

where the functions $S_0, S_1, S_2, ...$ satisfy the following equations

$$\begin{aligned}
S(u_0(\bar{x},t) + p\ u_1(\bar{x},t) + p^2 u_2(\bar{x},t) + ...) &= S_0(u_0(\bar{x},t)) + p S_1(u_0(\bar{x},t), u_1(\bar{x},t))\\
&+ p^2 S_2(u_0(\bar{x},t), u_1(\bar{x},t), u_2(\bar{x},t)) + ...
\end{aligned} \tag{13}$$

Applying the operator I_t^{α} on both sides of Equation (12) and considering the initial and boundary conditions, the terms of the series solution can be given by

$$\begin{aligned}
&u_0(\bar{x},t) = \sum_{i=0}^{n-1} \frac{c_i t^i}{i\,!},\\
&u_1(\bar{x},t) = -J_t^{\alpha}\ [\ Lu_0(\bar{x},t)] - J_t^{\alpha}[\ S_0(u_0(\bar{x},t))\ ] + J_t^{\alpha}[f\ (\bar{x},t)\ ],\\
&u_j(\bar{x},t) = -J_t^{\alpha}\ [\ Lu_{j-1}(\bar{x},t)] - J_t^{\alpha}[\ S_{j-1}(u_0(\bar{x},t), u_1(\bar{x},t), ..., u_{j-1}(\bar{x},t))\ ], \qquad j = 2,3,...
\end{aligned} \tag{14}$$

On setting $p = 1,$ we get an accurate approximation solution in the following form

$$u(\bar{x},t) = \sum_{i=0}^{\infty} u_i\ (\bar{x},t). \tag{15}$$

THE HOMOTOPY ANALYSIS METHOD (HAM)

To describe the basic ideas of the HAM, we consider the following differential equation

$$N\ [\ D_t^{\alpha} u(x,y,t)] = 0, \tag{16}$$

where N is a nonlinear operator for this problem, while D_t^{α} stand for the fractional derivative, x, y and t denotes independent variables and $u(x,y,t)$ is an unknown function.

By means of the HAM, one first construct zero-order deformation equation

$$(1-q)\ \ell(\phi(x,y,t;q) - u_0(x,y,t)) = qhH(t)N\ [\phi(x,y,t;q)], \tag{17}$$

where $q \in [0, 1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, ℓ is an auxiliary linear operator and $u_0(x,y,t)$ is an initial guess. Obviously, when $q = 0$ and $q = 1,$ it holds

$$\phi(x,y,t;0) = u_0(x,y,t), \qquad \phi(x,y,t;1) = u(x,y,t). \tag{18}$$

Liao (1992, 1995) expanded $\phi(x,y,t;q)$ in Taylor series with respect to the embedding parameter *q*, as follows:

$$\phi(x,y,t;q) = u_0(x,y,t) + \sum_{m=1}^{\infty} u_m(x,y,t) q^m, \tag{19}$$

where

$$u_m(x,y,t) = \frac{1}{m\,!} \frac{\partial^m \phi(x,y,t;q)}{\partial q^m}\bigg|_{q=0}. \tag{20}$$

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter *h* and the auxiliary function $H(t)$ are selected such that the series (19) is convergent at $q = 1,$ then we have from (19)

$$u(x,y,t) = u_0(x,y,t) + \sum_{m=1}^{\infty} u_m(x,y,t). \tag{21}$$

Let us define the vector

$$\vec{u}_n(t) = \{u_0(x,y,t), u_1(x,y,t), u_2(x,y,t), \ldots, u_n(x,y,t)\}. \tag{22}$$

Differentiating (17) m times with respect to q, then setting $q = 0$ and dividing then by $m!$, we have the mth-order deformation equations

$$\ell(\, u_m(x,y,t) - \chi_m u_{m-1}(x,y,t)\,) = h\, H(t)\, \Re_m(\vec{u}_{m-1}), \tag{23}$$

Where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x,y,t;q)]}{\partial q^{m-1}} \bigg|_{q=0}, \tag{24}$$

and

$$\chi_m = \begin{cases} 0 & m \le 1, \\ 1 & m > 1. \end{cases} \tag{25}$$

Applying the Riemann-Liouville integral operator J^{α} on both side of (23), we have

$$u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) - \chi_m \sum_{i=0}^{n-1} u_{m-1}^{i}(0^+) \frac{t^i}{i!} + h\, H(t) J^{\alpha} \Re_m(\vec{u}_{m-1}). \tag{26}$$

APPLICATIONS

Here, we use the homotopy perturbation and homotopy analysis methods to calculate the approximate solution of the fractional Zakharov-Kuznetsov equation. To calculate fractional derivative to hyperbolic function *sinh* will we use the fractional derivative of the exponential function which defined in Miller and Sugden (2009) as the follows

$$D_x^{\mu}(e^{\sigma x}) = \sigma^{\mu} e^{\sigma x}, \quad \sigma > 0, \qquad D_x^{\mu}(e^{\sigma x}) = (-1)^{\mu} (-\sigma)^{\mu} e^{\sigma x}, \quad \sigma < 0,$$

so that

$$D_x^{\alpha}[\, \sinh(b\,x)\,] = D_x^{\alpha}[\frac{1}{2}(e^{bx} - e^{-bx})] = \frac{1}{2}[\, b^{\alpha} e^{bx} - (-1)^{\alpha}\, b^{\alpha} e^{-bx}\,], \quad b > 0.$$

Example 1

Consider the fractional Zakharov-Kuznetsov equation in the following form

$$D_t^{\alpha} u + D_x^{\beta}(u^2) + \frac{1}{8} D_x^{3\beta}(u^2) + \frac{1}{8} D_x^{\beta} D_y^{2\gamma}(u^2) = 0, \quad t > 0, \quad 0 < \alpha, \beta, \gamma \le 1, \tag{27}$$

subject to the following initial conditions

$$u(x,y,0) = \frac{4}{3} \lambda \sinh^2(x+y), \tag{28}$$

where λ is an arbitrary constant.

By the homotopy perturbation technique, we construct a homotopy function $H(V, p)$ which satisfies

$$H(V,p) = (1-p)[D_t^{\alpha} V - D_t^{\alpha} V_0] + p[D_t^{\alpha} V + D_x^{\beta}(V^2) + \frac{1}{8} D_x^{3\beta}(V^2) + \frac{1}{8} D_x^{\beta} D_y^{2\gamma}(V^2)] = 0. \tag{29}$$

According to the homotopy perturbation method, we can first use the embedding parameter p as a small parameter, and assume that the solution of Equation (29) can be written as a power series in p as follows:

$$V(x,y,t) = V_0(x,y,t) + p\, V_1(x,y,t) + p^2 V_2(x,y,t) + p^3 V_3(x,y,t) + \ldots \tag{30}$$

Substituting Equation (30) into (29) and arranging the coefficients of powers of p, after some calculation we obtain

$$\begin{aligned}
&p^0 : D_t^{\alpha} V_0 = 0, \qquad V(x,y,0) = \frac{4}{3} \lambda \sinh^2(x+y), \\
&p^1 : D_t^{\alpha} V_1 + D_x^{\beta}(V_0^{\,2}) + \frac{1}{8} D_x^{3\beta}(V_0^{\,2}) + \frac{1}{8} D_x^{\beta} D_y^{2\gamma}(V_0^{\,2}) = 0, \\
&p^2 : D_t^{\alpha} V_2 + D_x^{\beta}(2V_0 V_1) + \frac{1}{8} D_x^{3\beta}(2V_0 V_1) + \frac{1}{8} D_x^{\beta} D_y^{2\gamma}(2V_0 V_1) = 0, \\
&p^3 : D_t^{\alpha} V_3 + D_x^{\beta}(2V_0 V_2 + V_1^{\,2}) + \frac{1}{8} D_x^{3\beta}(2V_0 V_2 + V_1^{\,2}) + \frac{1}{8} D_x^{\beta} D_y^{2\gamma}(2V_0 V_2 + V_1^{\,2}) = 0.
\end{aligned} \tag{31}$$

After some calculation, we have

$$V_0(x,y,t) = \frac{4}{3} \lambda \sinh^2(x+y), \tag{32}$$

$$\begin{aligned}
V_1(x,y,t) = \frac{-\lambda^2 t^{\alpha}}{9\,\Gamma(\alpha+1)} \Big\{ &[\, 2^{2\beta} + 2^{6\beta-3} + 2^{2\beta+4\gamma-3}][\, e^{4(x+y)} + (-1)^{\beta} e^{-4(x+y)}] \\
&- [\, 2^{\beta+2} + 2^{3\beta-1} + 2^{\beta+2\gamma-1}][\, e^{2(x+y)} + (-1)^{\beta} e^{-2(x+y)}] \Big\},
\end{aligned} \tag{33}$$

$$\begin{aligned}
V_2(x,y,t) = \frac{2\lambda^3 t^{2\alpha}}{27\Gamma(2\alpha+1)} \Big[&\, 2^{-5+2\beta}(8+4^{\beta}+4^{\gamma})(64+2^{3+\beta}+2^{3+2\beta}+2^{3+2\gamma}+2^{\beta+4\gamma}+32^{\beta})\cosh(2x+2y) \\
&- 2^{-4+3\beta}(8+16^{\beta}+16^{\gamma})(16+2^{3+\beta}+2^{1+2\beta}+2^{1+2\gamma}+2^{\beta+4\gamma}+32^{\beta})\cosh(4x+4y) \\
&+ 2^{-5+3\beta} 3^{\beta}(8+16^{\beta}+16^{\gamma})(8+32^{\beta}+32^{\gamma})\cosh(6x+6y) \Big],
\end{aligned} \tag{34}$$

$$V_3(x,y,t)=\frac{-\lambda^4 t^{3\alpha}\,\Gamma(2\alpha+1)}{81\,\Gamma(3\alpha+1)}\Big\{\omega_1[e^{2(x+y)}+(-1)^{\beta}e^{-2(x+y)}]+\omega_2[e^{4(x+y)}+(-1)^{\beta}e^{-4(x+y)}]$$
$$+\omega_3[e^{6(x+y)}+(-1)^{\beta}e^{-6(x+y)}]+\omega_4[e^{8(x+y)}+(-1)^{\beta}e^{-8(x+y)}]\Big\}, \tag{35}$$

and so on, where

$$\omega_1=-8^{-2+\beta}(8+4^{\beta}+4^{\gamma})\Big[\frac{(-2)^{\beta}(8+4^{\beta}+4^{\gamma})(8+16^{\beta}+16^{\gamma})}{\Gamma^2(\alpha+1)}$$
$$+\frac{2^{\beta}(8+16^{\beta}+16^{\gamma})(16+2^{3+\beta}+2^{1+2\beta}+2^{1+2\gamma}+2^{\beta+4\gamma}+32^{\beta})}{\Gamma(2\alpha+1)}$$
$$+\frac{(8+4^{\beta}+4^{\gamma})(64+2^{3+\beta}+2^{3+2\beta}+2^{3+2\gamma}+2^{\beta+4\gamma}+32^{\beta})}{\Gamma(2\alpha+1)}\Big], \tag{36}$$

$$\omega_2=2^{-7+4\beta}(8+16^{\beta}+16^{\gamma})\Big[\frac{6^{\beta}(8+16^{\beta}+16^{\gamma})(8+36^{\beta}+36^{\gamma})}{\Gamma(2\alpha+1)}$$
$$+\frac{4(8+4^{\beta}+4^{\gamma})^2}{\Gamma^2(\alpha+1)}+\frac{(8+4^{\beta}+4^{\gamma})(64+2^{3+\beta}+2^{3+2\beta}+2^{3+2\gamma}+2^{\beta+4\gamma}+32^{\beta})}{\Gamma(2\alpha+1)}$$
$$+\frac{2^{2+\beta}(8+16^{\beta}+16^{\gamma})(16+2^{3+\beta}+2^{1+2\beta}+2^{1+2\gamma}+2^{\beta+4\gamma}+32^{\beta})}{\Gamma(2\alpha+1)}\Big], \tag{37}$$

$$\omega_3=\frac{-3^{\beta}\;4^{-3+2\beta}(8+16^{\beta}+16^{\gamma})(8+36^{\beta}+36^{\gamma})}{\Gamma^2(\alpha+1)\,\Gamma(2\alpha+1)}\Big[(8+4^{\beta}+4^{\gamma})\Gamma(2\alpha+1)$$
$$+(16+2^{3+\beta}+2^{1+2\beta}+2^{1+2\gamma}+2^{\beta+4\gamma}+8\times 3^{\beta}+3^{\beta+2\gamma}\times 4^{\gamma}+32^{\beta}+108^{\beta})\Gamma^2(\alpha+1)\Big], \tag{38}$$

$$\omega_4=8^{-3+2\beta}(8+16^{\beta}+16^{\gamma})(8+64^{\beta}+64^{\gamma})\Big[\frac{2^{\beta}(8+16^{\beta}+16^{\gamma})}{\Gamma^2(\alpha+1)}+\frac{4\times 3^{\beta}(8+36^{\beta}+36^{\gamma})}{\Gamma(2\alpha+1)}\Big]. \tag{39}$$

On setting $p=1$, we get an accurate approximation solution by the homotopy perturbation method which takes the following form:

$$u(x,y,t)=\frac{4}{3}\lambda\sinh^2(x+y)-\frac{\lambda^2 t^{\alpha}}{9\Gamma(\alpha+1)}\Big\{[2^{2\beta}+2^{6\beta-3}+2^{2\beta+4\gamma-3}][e^{4(x+y)}+(-1)^{\beta}e^{-4(x+y)}]$$
$$-[2^{\beta+2}+2^{3\beta-1}+2^{\beta+2\gamma-1}][e^{2(x+y)}+(-1)^{\beta}e^{-2(x+y)}]\Big\}$$
$$+\frac{2\lambda^3 t^{2\alpha}}{27\Gamma(2\alpha+1)}\Big[2^{-5+2\beta}(8+4^{\beta}+4^{\gamma})(64+2^{3+\beta}+2^{3+2\beta}+2^{3+2\gamma}+2^{\beta+4\gamma}+32^{\beta})\cosh(2x+2y)$$
$$-2^{-4+3\beta}(8+16^{\beta}+16^{\gamma})(16+2^{3+\beta}+2^{1+2\beta}+2^{1+2\gamma}+2^{\beta+4\gamma}+32^{\beta})\cosh(4x+4y)$$
$$+2^{-5+3\beta}3^{\beta}(8+16^{\beta}+16^{\gamma})(8+32^{\beta}+32^{\gamma})\cosh(6x+6y)\Big]$$
$$-\frac{\lambda^4 t^{3\alpha}\,\Gamma(2\alpha+1)}{81\,\Gamma(3\alpha+1)}\Big\{\omega_1[e^{2(x+y)}+(-1)^{\beta}e^{-2(x+y)}]+\omega_2[e^{4(x+y)}+(-1)^{\beta}e^{-4(x+y)}]$$
$$+\omega_3[e^{6(x+y)}+(-1)^{\beta}e^{-6(x+y)}]+\omega_4[e^{8(x+y)}+(-1)^{\beta}e^{-8(x+y)}]\Big\}+\dots. \tag{40}$$

Equation (40) represented the approximate solution for the fractional Zakharov -Kuznetsov Equation (27) which was obtained by HPM.

By means of the homotopy analysis method, we choose the linear operator

$$\ell[\phi(x,y,t;q)]=\frac{\partial^{\alpha}\phi(x,y,t;q)}{\partial t^{\alpha}}, \tag{41}$$

with property $\ell[c]=0,$ where c is a constant. We define a nonlinear operator as

$$N[\phi(x,y,t;q)]=\frac{\partial^{\alpha}\phi(x,y,t;q)}{\partial t^{\alpha}}+\frac{\partial^{\beta}\phi^2(x,y,t;q)}{\partial x^{\beta}}+\frac{1}{8}\frac{\partial^{3\beta}\phi^2(x,y,t;q)}{\partial x^{3\beta}}+\frac{1}{8}\frac{\partial^{\beta}\partial^{2\gamma}\phi^2(x,y,t;q)}{\partial x^{\beta}\partial y^{2\gamma}}. \tag{42}$$

We consider auxiliary function $H(t)=1.$ So, the zeroth-order deformation equation

$$(1-q)\,\ell[\phi(x,y,t;q)-u_0(x,y,t)]=qhN[\phi(x,y,t;q)]. \tag{43}$$

For $q=0$ and $q=1,$ we can write

$$\phi(x,y,t;0)=u_0(x,y,t),\qquad \phi(x,y,t;1)=u(x,y,t). \tag{44}$$

Thus, we obtain the m^{th} -order deformation equations

$$u_m(x,y,t)=\chi_m u_{m-1}(x,y,t)+J_t^{\alpha}\Big[h\Big(D_t^{\alpha}u_{m-1}+D_x^{\beta}\sum_{n=0}^{m-1}u_n u_{m-1-n}+\frac{1}{8}D_x^{3\beta}\sum_{n=0}^{m-1}u_n u_{m-1-n}$$
$$+\frac{1}{8}D_x^{\beta}D_y^{2\gamma}\sum_{n=0}^{m-1}u_n u_{m-1-n}\Big)\Big],\quad m\ge 1. \tag{45}$$

By using the Equation (45), and after some calculation we obtain

$$u_1(x,y,t)=\frac{h\lambda^2 t^{\alpha}}{9\,\Gamma(\alpha+1)}\Big\{[2^{2\beta}+2^{6\beta-3}+2^{2\beta+4\gamma-3}][e^{4(x+y)}+(-1)^{\beta}e^{-4(x+y)}]$$
$$-[2^{\beta+2}+2^{3\beta-1}+2^{\beta+2\gamma-1}][e^{2(x+y)}+(-1)^{\beta}e^{-2(x+y)}]\Big\}, \tag{46}$$

$$u_2(x,y,t)=(h+1)u_1(x,y,t)+\frac{2h^2\lambda^3 t^{2\alpha}}{27\Gamma(2\alpha+1)}\Big[2^{-5+2\beta}(8+4^{\beta}+4^{\gamma})$$
$$(64+2^{3+\beta}+2^{3+2\beta}+2^{3+2\gamma}+2^{\beta+4\gamma}+32^{\beta})\cosh(2x+2y)$$
$$-2^{-4+3\beta}(8+16^{\beta}+16^{\gamma})(16+2^{3+\beta}+2^{1+2\beta}+2^{1+2\gamma}+2^{\beta+4\gamma}+32^{\beta})\cosh(4x+4y)$$
$$+2^{-5+3\beta}3^{\beta}(8+16^{\beta}+16^{\gamma})(8+32^{\beta}+32^{\gamma})\cosh(6x+6y)\Big], \tag{47}$$

$$u_3(x,y,t)=(h+1)u_2(x,y,t)+\frac{h^3\lambda^4 t^{3\alpha}\,\Gamma(2\alpha+1)}{81\,\Gamma(3\alpha+1)}\Big\{\omega_1[e^{2(x+y)}+(-1)^{\beta}e^{-2(x+y)}]$$
$$+\omega_2[e^{4(x+y)}+(-1)^{\beta}e^{-4(x+y)}]+\omega_3[e^{6(x+y)}+(-1)^{\beta}e^{-6(x+y)}]$$
$$+\omega_4[e^{8(x+y)}+(-1)^{\beta}e^{-8(x+y)}]\Big\}, \tag{48}$$

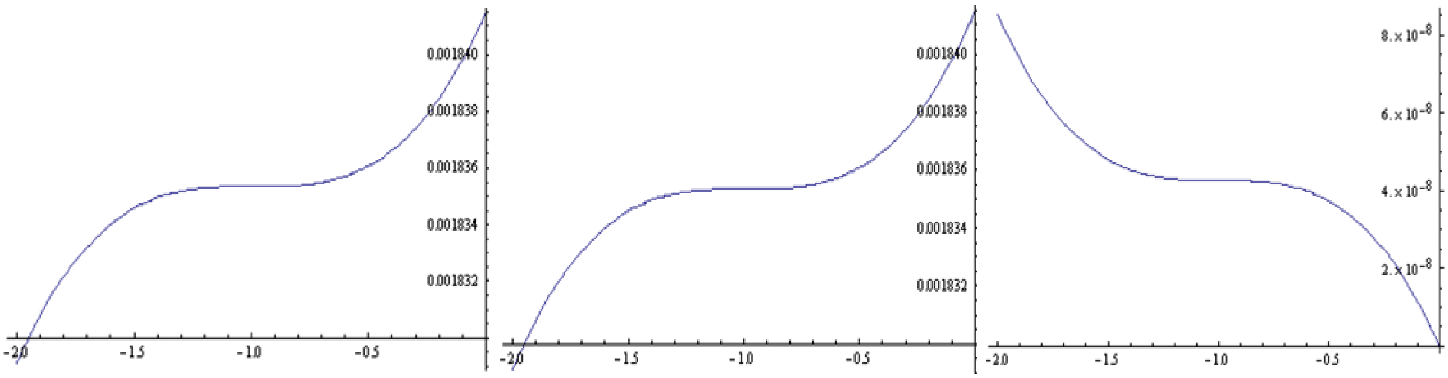


Figure 1. The h-curves of the four-order approximation to Abs $(u(x,y,t))$, Re $(u(x,y,t))$ and Im$(u(x,y,t))$ respectively at $\alpha=\beta=\gamma=0.5,\ \lambda=0.001,\ y=x=0.5,\ t=0.1$.

where $\omega_1,\omega_2,\omega_3,\omega_4$ take the same form (36)…(39) respectively.

In this case, the approximate solution by using the homotopy analysis method of Equation (27) is given by

$$
\begin{aligned}
u(x,y,t)=&\frac{4}{3}\lambda\sinh^2(x+y)+\frac{h\lambda^2t^\alpha}{9\Gamma(\alpha+1)}\Big\{[2^{2\beta}+2^{6\beta-3}+2^{2\beta+4\gamma-3}][e^{4(x+y)}+(-1)^\beta e^{-4(x+y)}]\\
&-[2^{\beta+2}+2^{3\beta-1}+2^{\beta+2\gamma-1}][e^{2(x+y)}+(-1)^\beta e^{-2(x+y)}]\Big\}\\
&+(h+1)u_1(x,y,t)+\frac{2h^2\lambda^3t^{2\alpha}}{27\Gamma(2\alpha+1)}\Big[2^{-5+2\beta}(8+4^\beta+4^\gamma)\\
&(64+2^{3+\beta}+2^{3+2\beta}+2^{3+2\gamma}+2^{\beta+4\gamma}+32^\beta)\cosh(2x+2y)\\
&-2^{-4+3\beta}(8+16^\beta+16^\gamma)(16+2^{3+\beta}+2^{1+2\beta}+2^{1+2\gamma}+2^{\beta+4\gamma}+32^\beta)\cosh(4x+4y)\\
&+2^{-5+3\beta}3^\beta(8+16^\beta+16^\gamma)(8+32^\beta+32^\gamma)\cosh(6x+6y)\Big]\\
&+(h+1)u_2(x,y,t)+\frac{h^3\lambda^4t^{3\alpha}\Gamma(2\alpha+1)}{81\Gamma(3\alpha+1)}\Big\{\omega_1[e^{2(x+y)}+(-1)^\beta e^{-2(x+y)}]\\
&+\omega_2[e^{4(x+y)}+(-1)^\beta e^{-4(x+y)}]+\omega_3[e^{6(x+y)}+(-1)^\beta e^{-6(x+y)}]\\
&+\omega_4[e^{8(x+y)}+(-1)^\beta e^{-8(x+y)}]\Big\}\\
&+\ldots
\end{aligned}
\tag{49}
$$

Equation (49) represented the approximate solution for the fractional Zakharov-Kuznetsov Equation (27) which was obtained by HAM.

Remarks 1

1) The homotopy analysis method determines the interval of convergence from the h-curve (Figure 1). As pointed by Liao (1992, 1995), the valid region of *h* is a horizontal line segment. Therefore, it is straightforward to choose an appropriate range for *h* which ensure the convergence of the solution series. We stretch the h-curve of $u(0.5,0.5,0.1)$ in Figure 1, which shows that the solution series is convergence when $-1.5<h<-0.5$.

2) In special case, when $\alpha,\beta,\gamma\to1$ in Equations (32) to (35), we get

$$
\begin{aligned}
V_0(x,y,t)=&\frac{4}{3}\lambda\sinh^2(x+y),\\
V_1(x,y,t)=&\frac{-224}{9}\lambda^2\sinh^3(x+y)\cosh(x+y)t-\frac{32}{3}\lambda^2\sinh(x+y)\cosh^3(x+y)t,\\
V_2(x,y,t)=&\frac{64}{27}\lambda^3(1200\cosh^6(x+y)-2080\cosh^4(x+y)+968\cosh^2(x+y)-79)t^2,\\
V_3(x,y,t)=&\frac{-4096}{243}\sinh(x+y)\cosh(x+y)\Big[23800\cosh^6(x+y)\\
&-42900\cosh^4(x+y)+22665\cosh^2(x+y)-3142\Big]\lambda^4t^3,
\end{aligned}
\tag{50}
$$

is the same solutions obtained by Biazar et al. (2009).

Table 1 leads to the absolute error between the approximate solutions (40) obtained by HPM and the approximate solutions (49) obtained by HAM. The absolute error is very small so that the approximate solutions has the same behavior.

Approximate solutions obtained by HPM tends to the approximate obtained by HAM when $h\to-1$.

The comparison between the approximate solution of Equation (27) by using the HPM and approximate solution by using HAM are shown in Figure 2.

Example 2

Consider the fractional Zakharov-Kuznetsov equation in the following form

$$
D_t^\alpha u-D_x^\beta(u^2)+\frac{1}{8}D_x^{3\beta}(u^2)+\frac{1}{8}D_x^\beta D_y^{2\gamma}(u^2)=0,\qquad t>0,\quad 0<\alpha,\beta,\gamma\le1,
\tag{51}
$$

subject to the following initial conditions (Hesam et al., 2012):

Table 1. Determine the absolute error between the real part of approximate solutions (40) which obtained by the HPM and the real part of approximate solutions (49) which obtained by the HAM when $y = \alpha = \beta = \gamma = 0.1,\ h = -1.1$ and $\lambda = 0.001$.

x	t	Re U $_{HPM}$	Re U $_{HAM}$	$\lvert u_{HPM} - u_{HAM} \rvert$
	0.1	0.5476301844E-4	0.5476378484E-4	0.76640E-9
0.1	0.5	0.5488796450E-4	0.5488887554E-4	0.91104E-9
	0.9	0.5493885905E-4	0.5493982991E-4	0.97086E-9
	0.1	0.5407093554E-3	0.5407096643E-3	0.3089E-9
0.5	0.5	0.5407569911E-3	0.5407573617E-3	0.3706E-9
	0.9	0.5407763977E-3	0.5407767940E-3	0.3963E-9
	0.1	0.1837593296E-2	0.1837591245E-2	0.2051E-8
0.9	0.5	0.1836921718E-2	0.1836919675E-2	0.2043E-8
	0.9	0.1836648586E-2	0.1836646577E-2	0.2009E-8

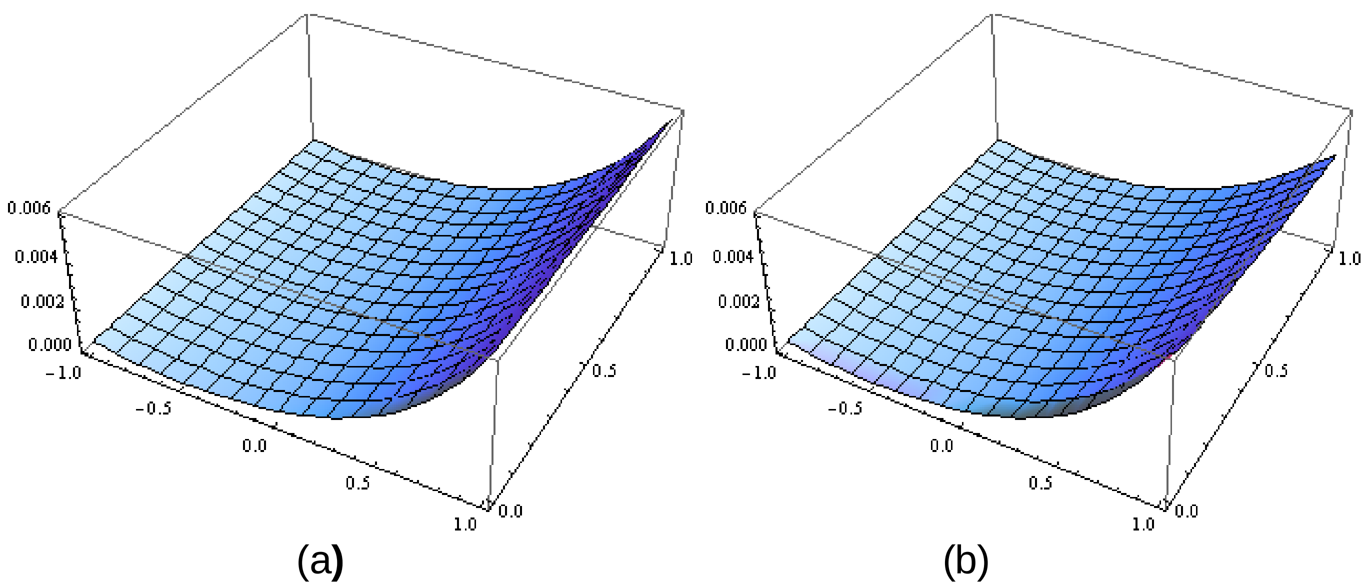


Figure 2. (a) Represents the real part of approximate solution (40) by HPM and (b) represents the real part of approximate solution (49) by HAM at $\alpha = \beta = \gamma = y = 0.5, \quad \lambda = 0.001, \quad h = -3, \ -1 \le x \le 1$ and $0 \le t \le 1$.

$$u(x, y, 0) = \frac{4}{3}\lambda \sinh^2\left[\frac{1}{2}(x + y)\right], \tag{52}$$

where λ is an arbitrary constant.

By the homotopy perturbation technique, we construct a homotopy which satisfies

$$H(V, p) = (1-p)[D_t^{\alpha}V - D_t^{\alpha}V_0] + p[D_t^{\alpha}V - D_x^{\beta}(V^2) + \frac{1}{8}D_x^{3\beta}(V^2) + \frac{1}{8}D_x^{\beta}D_y^{2\gamma}(V^2)] = 0. \tag{53}$$

According to the homotopy perturbation method, we can first use the embedding parameter p as a small parameter, and assume that the solution of Equation (53) can be written as a power series in p as follows:

$$V(x,y,t) = V_0(x,y,t) + pV_1(x,y,t) + p^2V_2(x,y,t) + p^3V_3(x,y,t) + \ldots \tag{54}$$

Substituting Equation (54) into (53) and arranging the coefficients of powers of p, after some calculation we obtain

$$p^0: D_t^{\alpha}V_0 = 0, \qquad V(x,y,0) = \frac{4}{3}\lambda \sinh^2[\frac{1}{2}(x+y)],$$

$$p^1: D_t^{\alpha}V_1 - D_x^{\beta}(V_0^2) + \frac{1}{8}D_x^{3\beta}(V_0^2) + \frac{1}{8}D_x^{\beta}D_y^{2\gamma}(V_0^2) = 0,$$

$$p^2: D_t^{\alpha}V_2 - D_x^{\beta}(2V_0V_1) + \frac{1}{8}D_x^{3\beta}(2V_0V_1) + \frac{1}{8}D_x^{\beta}D_y^{2\gamma}(2V_0V_1) = 0,$$

$$p^3: D_t^{\alpha}V_3 - D_x^{\beta}(2V_0V_2 + V_1^2) + \frac{1}{8}D_x^{3\beta}(2V_0V_2 + V_1^2) + \frac{1}{8}D_x^{\beta}D_y^{2\gamma}(2V_0V_2 + V_1^2) = 0. \tag{55}$$

After some calculation, we have

$$V_0(x,y,t) = \frac{4}{3}\lambda \sinh^2[\frac{1}{2}(x+y)], \tag{56}$$

$$V_1(x,y,t) = \frac{-\lambda^2 t^{\alpha}}{9\,\Gamma(\alpha+1)}\{[-2^{\beta} + 2^{3\beta-3} + 2^{\beta+2\gamma-3}][e^{2(x+y)} + (-1)^{\beta}e^{-2(x+y)}] + 3e^{(x+y)} + 3(-1)^{\beta}e^{-(x+y)}]\}, \tag{57}$$

$$V_2(x,y,t) = \frac{2\lambda^3 t^{2\alpha}}{27\Gamma(2\alpha+1)}[\ 2^{-5+\beta}3^{\beta}(-8+4^{\beta}+4^{\gamma})(-8+9^{\beta}+9^{\gamma})\cosh(3x+3y) - 2^{-4+\beta}(-8+4^{\beta}+4^{\gamma})(-12-2^{3+\beta}+2^{\beta+2\gamma}+8^{\beta})\cosh(2x+2y) - \frac{3}{16}(-48-2^{3+\beta}+2^{\beta+2\gamma}+8^{\beta})\cosh(x+y)\ ], \tag{58}$$

$$V_3(x,y,t) = \frac{-\lambda^4 t^{3\alpha}\Gamma(2\alpha+1)}{81\,\Gamma(3\alpha+1)}\{\rho_1[e^{4(x+y)} + (-1)^{\beta}e^{-4(x+y)}] + \rho_2[e^{3(x+y)} + (-1)^{\beta}e^{-3(x+y)}] + \rho_3[e^{2(x+y)} + (-1)^{\beta}e^{-2(x+y)}] + \rho_4[e^{(x+y)} + (-1)^{\beta}e^{-(x+y)}]\}, \tag{59}$$

and so on, where

$$\rho_1 = (-2^{2+\beta} + 2^{-3+6\beta} + 2^{-3+2\beta+4\gamma})\ [\ \frac{4(-2^{\beta}+2^{3\beta-3}+2^{\beta+2\gamma-3})(-3^{\beta}+\frac{3^{3\beta}}{8}+\frac{3^{\beta+2\gamma}}{8})}{\Gamma(2\alpha+1)} + \frac{(-2^{\beta}+2^{3\beta-3}+2^{\beta+2\gamma-3})^2}{\Gamma^2(\alpha+1)}\ ], \tag{60}$$

$$\rho_2 = \frac{2^{-6+\beta}3^{\beta}(-8+4^{\beta}+4^{\gamma})(-8+9^{\beta}+9^{\gamma})}{\Gamma(2\alpha+1)\ \Gamma^2(\alpha+1)}\ [\ 6\,\Gamma(2\alpha+1) - (-12-2^{3+\beta}+2^{\beta+2\gamma}-8\times 3^{\beta}+3^{\beta+2\gamma}+8^{\beta}+27^{\beta})\Gamma^2(\alpha+1)\ ], \tag{61}$$

$$\rho_3 = 2^{-3+\beta}(-8+4^{\beta}+4^{\gamma})[\frac{9}{\Gamma^2(\alpha+1)} + \frac{4}{\Gamma(2\alpha+1)}(\frac{9}{2} + 27\times 2^{-2+\beta} - 27\times 2^{-5+3\beta} - 27\times 2^{-5+\beta+2\gamma} + 4^{-2+\beta}(-8+4^{\beta}+4^{\gamma})^2 + 2^{-6+\beta}\times 3^{\beta}(-8+4^{\beta}+4^{\gamma})(-8+9^{\beta}+9^{\gamma})], \tag{62}$$

$$\rho_4 = -\frac{3}{4}[\frac{-288-9\times 2^{4+\beta}+9\times 2^{1+3\beta}-4^{3+\beta}-4^{\beta+2\gamma}+16^{1+\beta}-64^{\beta}+2^{1+\beta+2\gamma}(9+2^{3+\beta}-8^{\beta})}{8\,\Gamma(2\alpha+1)} + \frac{6(-1)^{\beta}(-2^{\beta}+2^{-3+3\beta}+2^{-3+\beta+2\gamma})}{\Gamma^2(\alpha+1)}]. \tag{63}$$

On setting $p=1$, we get an accurate approximation solution by the homotopy perturbation method which takes the following form:

$$u(x,y,t) = \frac{4}{3}\lambda \sinh^2[\frac{1}{2}(x+y)] - \frac{\lambda^2 t^{\alpha}}{9\,\Gamma(\alpha+1)}\{[-2^{\beta}+2^{3\beta-3}+2^{\beta+2\gamma-3}][e^{2(x+y)}+(-1)^{\beta}e^{-2(x+y)}] + 3e^{(x+y)} + 3(-1)^{\beta}e^{-(x+y)}]\}$$
$$+ \frac{2\lambda^3 t^{2\alpha}}{27\Gamma(2\alpha+1)}\{\ 2^{-5+\beta}3^{\beta}(-8+4^{\beta}+4^{\gamma})(-8+9^{\beta}+9^{\gamma})\cosh(3x+3y) - 2^{-4+\beta}(-8+4^{\beta}+4^{\gamma})(-12-2^{3+\beta}+2^{\beta+2\gamma}+8^{\beta})\cosh(2x+2y) - \frac{3}{16}(-48-2^{3+\beta}+2^{\beta+2\gamma}+8^{\beta})\cosh(x+y)\ \}$$
$$- \frac{\lambda^4 t^{3\alpha}\Gamma(2\alpha+1)}{81\,\Gamma(3\alpha+1)}\{\rho_1[e^{4(x+y)}+(-1)^{\beta}e^{-4(x+y)}] + \rho_2[e^{3(x+y)}+(-1)^{\beta}e^{-3(x+y)}] + \rho_3[e^{2(x+y)}+(-1)^{\beta}e^{-2(x+y)}] + \rho_4[e^{(x+y)}+(-1)^{\beta}e^{-(x+y)}]\} + \ldots \tag{64}$$

Equation (64) represented the approximate solution for the fractional Zakharov -Kuznetsov Equation (51 obtained by HPM.

By means of the homotopy analysis method, we choose the linear operator

$$\ell[\phi(x,y,t;q)] = \frac{\partial^{\alpha}\phi(x,y,t;q)}{\partial t^{\alpha}}, \tag{65}$$

with property $\ell[c]=0,$ where c is a constant. We define a nonlinear operator as

$$N[\phi(x,y,t;q)] = \frac{\partial^{\alpha}\phi(x,y,t;q)}{\partial t^{\alpha}} - \frac{\partial^{\beta}\phi^2(x,y,t;q)}{\partial x^{\beta}} + \frac{1}{8}\frac{\partial^{3\beta}\phi^2(x,y,t;q)}{\partial x^{3\beta}} + \frac{1}{8}\frac{\partial^{\beta}\partial^{2\gamma}\phi^2(x,y,t;q)}{\partial x^{\beta}\partial y^{2\gamma}}. \tag{66}$$

We consider auxiliary function $H(t)=1.$ So, the zeroth-order deformation equation

$$(1-q)\ \ell[\ \phi(x,y,t;q) - u_0(x,y,t)\ ] = qhN[\phi(x,y,t;q)]. \tag{67}$$

For $q=0$ and $q=1,$ we can write

$$\phi(x,y,t;0) = u_0(x,y,t), \qquad \phi(x,y,t;1) = u(x,y,t). \tag{68}$$

Thus, we obtain the *m*th-order deformation equations

$$u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + J_t^{\alpha}\Big[h\Big(D_t^{\alpha}u_{m-1} - D_x^{\beta}\sum_{n=0}^{m-1}u_n u_{m-1-n} + \frac{1}{8}D_x^{3\beta}\sum_{n=0}^{m-1}u_n u_{m-1-n} + \frac{1}{8}D_x^{\beta}D_y^{2\gamma}\sum_{n=0}^{m-1}u_n u_{m-1-n}\Big)\Big], \quad m \ge 1. \tag{69}$$

Table 2. Determine the absolute error between the real part of approximate solutions (64) obtained by the HPM and the real part of approximate solutions (73) obtained by the HAM when $y=\alpha=\beta=\gamma=0.1,\ h=-1.1$ and $\lambda=0.001$.

x	t	Re U $_{HPM}$	Re U $_{HAM}$	$\lvert u_{HPM}-u_{HAM}\rvert$
0.1	0.1	0.1297232493E-4	0.1297193845E-4	0.38648E-9
	0.5	0.1290155925E-4	0.1290110921E-4	0.45004E-9
	0.9	0.1287274434E-4	0.1287226875E-4	0.47559E-9
0.5	0.1	0.1232468684E-3	0.1232464860E-3	0.3824E-9
	0.5	0.1231776456E-3	0.1231771992E-3	0.4464E-9
	0.9	0.1231494580E-3	0.1231489857E-3	0.4723E-9
0.9	0.1	0.3617327474E-3	0.3617324328E-3	0.3146E-9
	0.5	0.3616767081E-3	0.3616763399E-3	0.3682E-9
	0.9	0.3616538879E-3	0.3616534979E-3	0.3900E-9

By using the Equation (69), and after some calculation we obtain

$$u_1(x,y,t)=\frac{h\,\lambda^2 t^{\alpha}}{9\,\Gamma(\alpha+1)}\Big\{[-2^{\beta}+2^{3\beta-3}+2^{\beta+2\gamma-3}]\,[e^{2(x+y)}+(-1)^{\beta}\,e^{-2(x+y)}] \quad (70)$$
$$+3e^{(x+y)}+3(-1)^{\beta}e^{-(x+y)}]\Big\},$$

$$u_2(x,y,t)=(h+1)u_1(x,y,t)+\frac{2h^2\lambda^3 t^{2\alpha}}{27\Gamma(2\alpha+1)}\Big[\,2^{-5+\beta}3^{\beta}(-8+4^{\beta}+4^{\gamma})(-8+9^{\beta}+9^{\gamma})\cosh(3x+3y)$$
$$-2^{-4+\beta}(-8+4^{\beta}+4^{\gamma})(-12-2^{3+\beta}+2^{\beta+2\gamma}+8^{\beta})\cosh(2x+2y)$$
$$-\frac{3}{16}(-48-2^{3+\beta}+2^{\beta+2\gamma}+8^{\beta})\cosh(x+y)\ \Big], \quad (71)$$

$$u_3(x,y,t)=(h+1)u_2(x,y,t)+\frac{h^3\lambda^4 t^{3\alpha}\Gamma(2\alpha+1)}{81\,\Gamma(3\alpha+1)}\Big\{\rho_1[e^{4(x+y)}+(-1)^{\beta}e^{-4(x+y)}]+$$
$$\rho_2[e^{3(x+y)}+(-1)^{\beta}e^{-3(x+y)}]+\rho_3[e^{2(x+y)}+(-1)^{\beta}e^{-2(x+y)}]+\rho_4[e^{(x+y)}+(-1)^{\beta}e^{-(x+y)}]\Big\}, \quad (72)$$

$$\vdots$$

where $\rho_1,\rho_2,\rho_3,\rho_4$ take the same form (60)…(63) respectively. In this case, the approximate solution by using HAM of Equation (51) is given by

$$u(x,y,t)=\frac{4}{3}\lambda\sinh^2[\frac{1}{2}(x+y)]+\frac{h\lambda^2 t^{\alpha}}{9\,\Gamma(\alpha+1)}\Big\{[-2^{\beta}+2^{3\beta-3}+2^{\beta+2\gamma-3}]\,[e^{2(x+y)}+(-1)^{\beta}\,e^{-2(x+y)}]$$
$$+3e^{(x+y)}+3(-1)^{\beta}e^{-(x+y)}]\Big\}$$
$$+(h+1)u_1(x,y,t)+\frac{2h^2\lambda^3 t^{2\alpha}}{27\Gamma(2\alpha+1)}\Big[\,2^{-5+\beta}3^{\beta}(-8+4^{\beta}+4^{\gamma})(-8+9^{\beta}+9^{\gamma})\cosh(3x+3y)$$
$$-2^{-4+\beta}(-8+4^{\beta}+4^{\gamma})(-12-2^{3+\beta}+2^{\beta+2\gamma}+8^{\beta})\cosh(2x+2y)$$
$$-\frac{3}{16}(-48-2^{3+\beta}+2^{\beta+2\gamma}+8^{\beta})\cosh(x+y)\ \Big]$$
$$+(h+1)u_2(x,y,t)+\frac{h^3\lambda^4 t^{3\alpha}\Gamma(2\alpha+1)}{81\,\Gamma(3\alpha+1)}\Big\{\rho_1[e^{4(x+y)}+(-1)^{\beta}e^{-4(x+y)}]+$$
$$\rho_2[e^{3(x+y)}+(-1)^{\beta}e^{-3(x+y)}]+\rho_3[e^{2(x+y)}+(-1)^{\beta}e^{-2(x+y)}]+\rho_4[e^{(x+y)}+(-1)^{\beta}e^{-(x+y)}]\Big\}$$
$$+\dots. \quad (73)$$

Equation (73) represented the approximate solution for the fractional Zakharov-Kuznetsov Equation (51) obtained by HAM.

Remarks 2

1) The homotopy analysis method determines the interval of convergence from the h-curve (Figure 5). As pointed by Liao (1992), the valid region of *h* is a horizontal line segment. Therefore, it is straightforward to choose an appropriate range for *h* which ensures the convergence of the solution series. We stretch the h-curve of $u(0.1,0.1,0.2)$ in Figure 5, which shows that the solution series is convergence when $-1.5<h<-0.5$.

2) In special case, when $\alpha,\beta,\gamma\to 1$, the approximate solution (64) takes the following form

$$u_{app}(x,y,t)=\frac{4}{3}\lambda\sinh^2[\frac{1}{2}(x+y)]-\frac{2}{3}\lambda^2 t\sinh(x+y)+\frac{1}{3}\lambda^3 t^2\cosh(x+y)-\frac{1}{9}\lambda^4 t^3\sinh(x+y)+\dots. \quad (74)$$

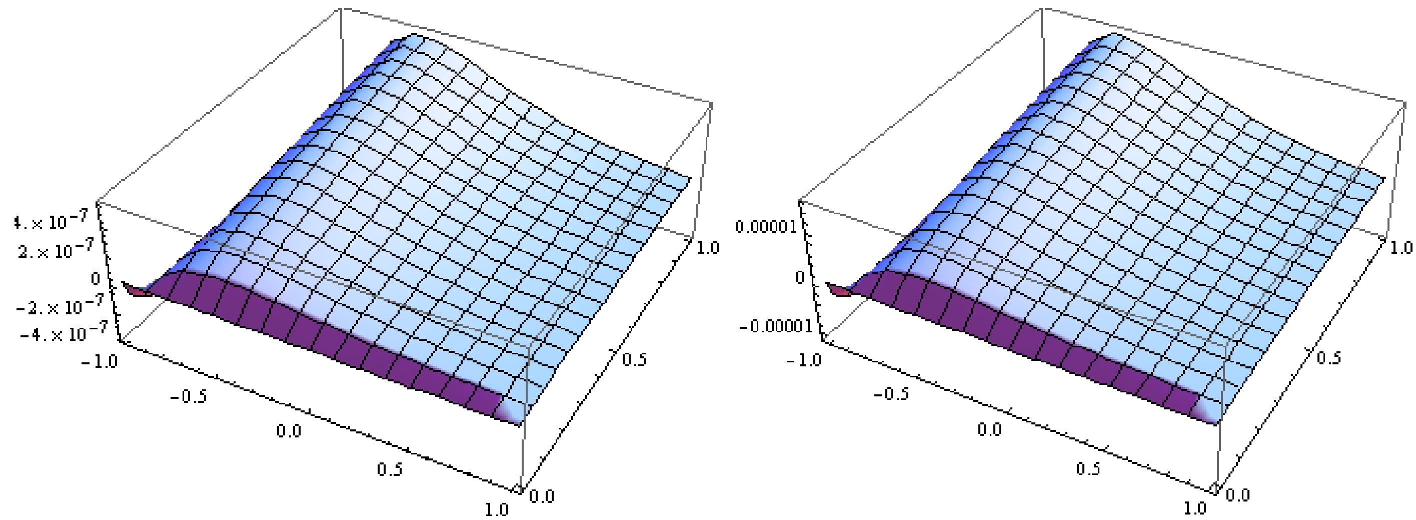
Using Taylor series expansion near $t=0,$ we get

$$u_{ex}(x,y,t)=\frac{4}{3}\lambda\sinh^2[\frac{1}{2}(x+y-\lambda t)]. \quad (75)$$

This solution (75) is exactly the same solution obtained in (Hesam et al., 2012).

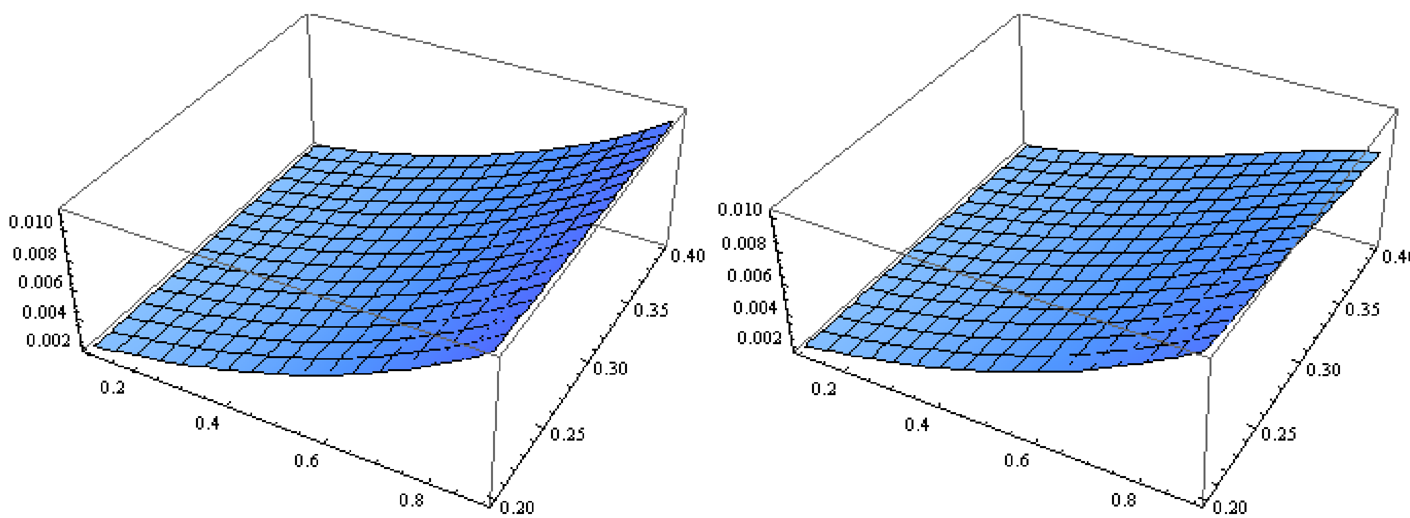
Table 2 leads to the absolute error between the approximate solutions (64) obtained by HPM and the approximate solutions (73) obtained by HAM. The absolute error is very small so that the approximate solutions have the same behavior.

The comparison between the approximate solution of Equation (51) by using the HPM and approximate solution by using HAM are shown in Figure 3, 4, 6, 7, 8.



(a) (b)

Figure 3. (a) Represents the imaginary part of approximate solution (40) by HPM and (b) represents the imaginary part of approximate solution (49) by HAM at $\alpha = \beta = \gamma = y = 0.5$, $\lambda = 0.001$, $h = -3$, $-1 \le x \le 1$ and $0 \le t \le 1$.



(a) (b)

Figure 4. (a) Represents approximate solution (40) by HPM and (b) represents approximate solution (49) by HAM at $\alpha = \beta = \gamma = 1$, $\lambda = 0.001$, $y = 0.9$, $h = -2$, $0.1 \le x \le 0.9$ and $0.2 \le t \le 0.4$.

Conclusion

In this paper, we used the two different methods such as homotopy perturbation method and homotopy analysis method to obtain analytic approximate solutions for the fractional Zakharov-Kuznetsov equations which are very important in mathematical physics especially in nonlinear dynamics and plasma physics. We compared between the approximate solutions obtained by using the homotopy perturbation method and the approximate solutions obtained by using the homotopy analysis method. The homotopy analysis method investigates the influence of h on the convergence of the approximate solution. Note that the solution series contains the auxiliary parameter h which provides us with a simple way to adjust and control the convergence of the solution series. Also we compared between the approximate solutions obtained by these methods and the exact solutions when $\alpha, \beta, \gamma \to 1$. These methods are effective and allows us to solve nonlinear partial fractional

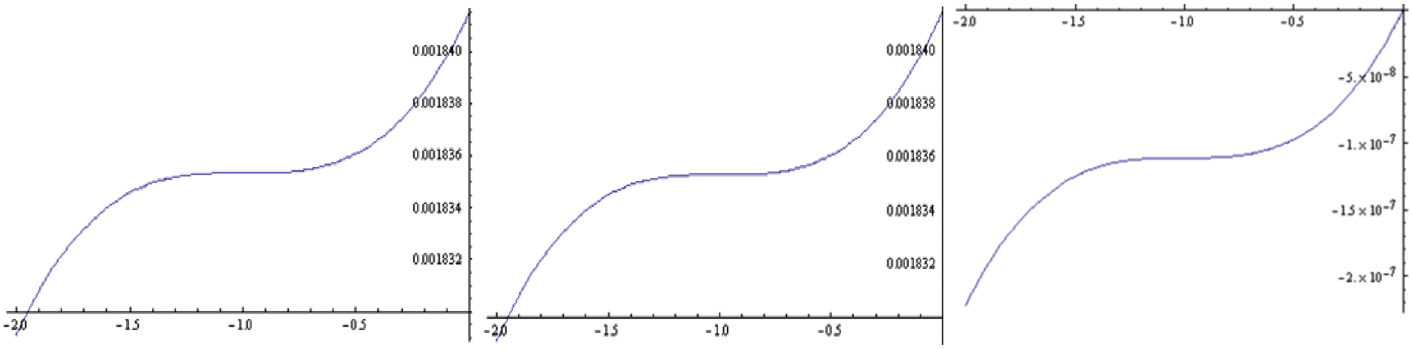
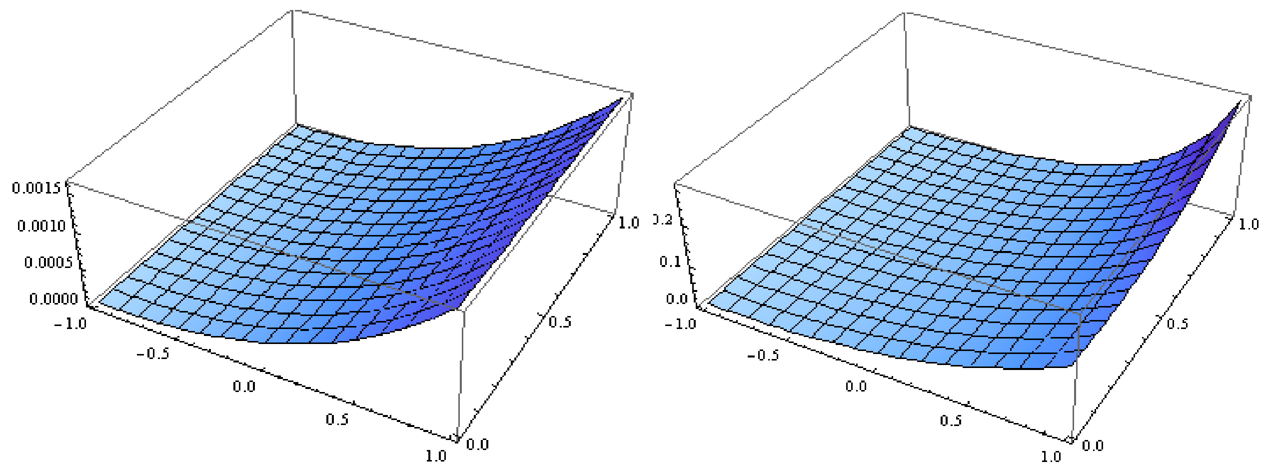
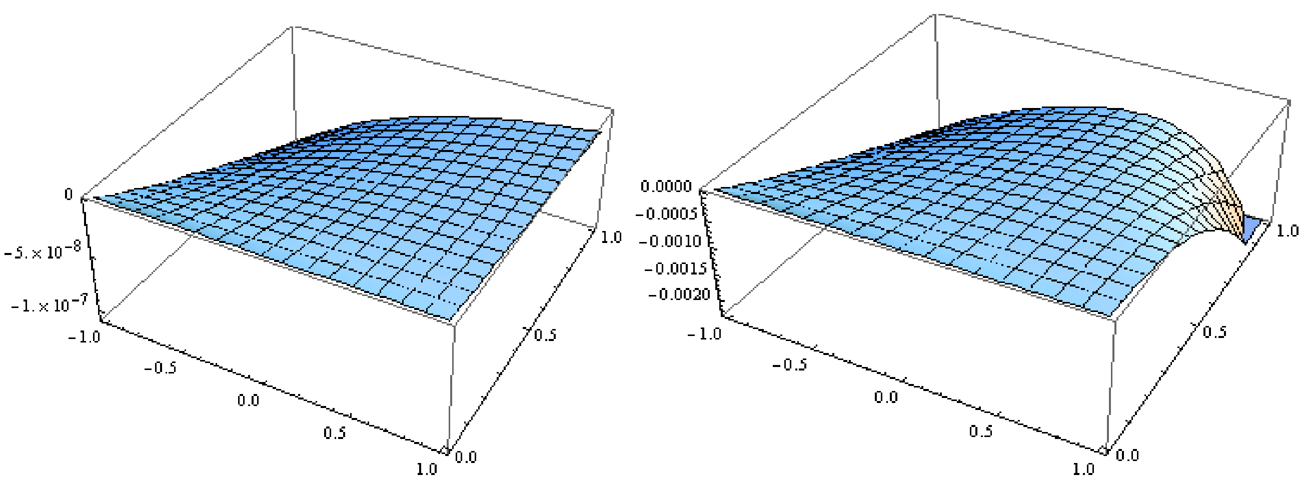


Figure 5. The h-curves of the four-order approximation to Abs $(u(x,y,t))$, Re$(u(x,y,t))$ and Im$(u(x,y,t))$ respectively at $\alpha=\beta=\gamma=0.5,\ \lambda=0.001,\ y=x=0.1,\ t=0.2$.



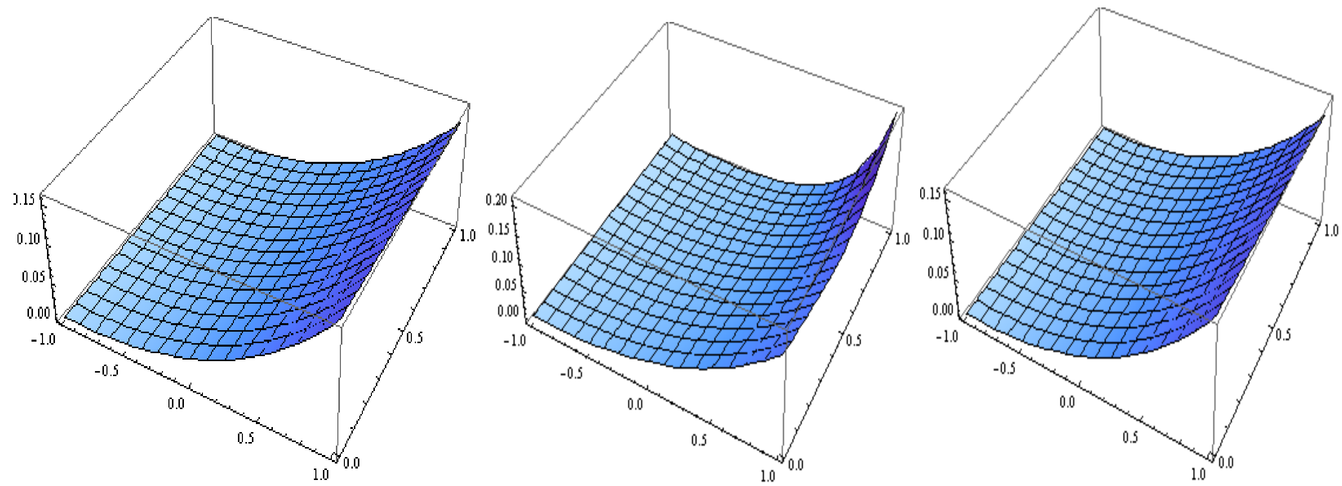
(a) (b)

Figure 6. (a) Represents the real part of approximate solution (64) by HPM and (b) represents the real part of approximate solution (73) by HAM at $\alpha=\beta=\gamma=0.9,\ \ y=0.1,\ \ \lambda=0.1,\ \ -1\le x\le 1$ and $0\le t\le 1$.



(a) (b)

Figure 7. (a) Represents the imaginary part of approximate solution (64) by HPM and (b) represents the imaginary part of approximate solution (73) by HAM at $\alpha=\beta=\gamma=0.9,\ \ y=0.1,\ \ \lambda=0.1,\ h=-2,\ \ -1\le x\le 1$ and $0\le t\le 1$.



(a) (b) (c)

Figure 8. (a) Represents approximate solution (64) by HPM, (b) represents approximate solution (73) by HAM and (c) represents the exact solution (75) at $\alpha=\beta=\gamma=1,\ \lambda=0.1,\ y=0.9,\ h=-2.5,\ -1\le x\le 1$ and $0\le t\le 1$.

differential equations.

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