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Chebyshev wavelets method for boundary value problems

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Chebyshev wavelets method (CWM) is applied to find numerical solutions of fifth and sixth order boundary value problems. Computational work is fully supportive of compatibility of proposed algorithm and hence the same may be extended to other physical problems also. A very high level of accuracy explicitly reflects the reliability of this scheme for such problems.

Key words: Chebyshev wavelets method (CWM), boundary value problems, linear and nonlinear problems, exact solutions.

INTRODUCTION

Wavelet theory (Babolian and Fattah, 2007; Cattani and Kudreyko, 2010; Dehghan and Saadatmandi, 2008; Mohammadi et al., 2011; Maleknejad and Mirzaee, 2005; Rawashdeh, 2011; Razzaghi and Yousefi, 2002; Yousefi and Banifatemi, 2006) is one of the relatively new techniques which is being utilized for solving wide range of physical problems related to various branches of engineering and applied sciences. With the passage of time, lot of rapid developments is taking place which are helpful in increasing the accuracy of this scheme. The most common related schemes are Haar wavelets (Maleknejad and Mirzaee, 2005), harmonic wavelets of successive approximation (Cattani and Kudreyko, 2010), CAS wavelets (Yousefi and Banifatemi, 2006), Legendre Wavelets (Mohammadi et al., 2011; Rawashdeh, 2011; Razzaghi and Yousefi, 2002; Mohammadi and Hosseini, 2010) and Chebyshev wavelets (Babolian and Fattah, 2007; Dehghan and Saadatmandi, 2008). In the similar context, we merge Chebyshev polynomials with the traditional wavelet technique. The modified version which is called Chebyshev wavelets method (CWM) proves to be fully compatible with the complexity of the given problems and obtained results are extremely accurate. In particular, we apply CWM on linear and nonlinear boundary value problems of fifth and sixth orders. It is worth mentioning that such fifth order equations arise in the mathematical modeling of viscoelastic flows (Davis et al., 1988a; Davis et al., 1988b) and sixth order equations (Boutayeb and Twizell, 1992; Wazwaz, 2000) are known to arise in astrophysics (Toomore et al., 1976); the narrow convecting layers bounded by stable layers which are believed to surround A-type stars (Siddique and Ghazala, 2008; Twizell and Boutayeb, 1990). It is observed that CWM is very user friendly but is extremely accurate. The error estimates explicitly reveal the very high accuracy level of the suggested technique. It is to be highlighted that, recently, Yang presented local fractional wavelet transform method (Yang, 2011) which is extremely useful for linear and nonlinear problems of fractional order.

PROPERTIES OF SECOND CHEBYSHEV WAVELETS

Wavelets constitute a family of functions constructed from dilation and translation of a single function $\psi(x)$ called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously we have the following family of continuous wavelets as

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[10] $\psi_{a,b}(x)=|a|^{-\frac{1}{2}}\psi\left(\frac{x-b}{a}\right),\ a,\ b\in R,\ a\neq 0.$

If we restrict the parameters a and b to discrete values as $a=a_0^{-k},\ b=nb_0a_0^{-k},\ a_0>1,\ b_0>0,$ we have the following family of discrete wavelets

$$\psi_{k,n}(x)=|a|^{\frac{k}{2}}\psi\left(a_0{}^k x-nb_0\right),\ k,n\in$$

where $\psi_{k,n}$ form a wavelet basis for $L^2(R)$. In particular, when $a_0=2$ and $b_0=1$, then $\psi_{k,n}(x)$ form an orthonormal basis.

The second Chebyshev wavelets $\psi_{n,m}(x)=\psi(k,n,m,x)$ involve four arguments $n=1,2,\cdots,2^{k-1},\ k$ is assumed any positive integer, m is the degree of the second Chebyshev polynomials and it is the normalized time. They are defined on the interval $[0,1)$ as

$$\psi_{n,m}(x)=\begin{cases}2^{\frac{k}{2}}\tilde{T}_m\left(2^k x-2n+1\right), & \frac{n-1}{2^{k-1}}\leq x<\frac{n}{2^{k-1}},\\ 0, & \text{otherwise}\end{cases} \tag{1}$$

where $$\tilde{T}_m(x)=\sqrt{\frac{2}{\pi}}T_m(x), \tag{2}$$

$m=0,1,2,\cdots,M-1$. In Equation (2) the coefficients are used for orthonormality. Here $T_m(x)$ are the second Chebyshev polynomials of degree m with respect to the weight function $w(x)=\sqrt{1-x^2}$ on the interval $[-1,1]$, and satisfy the following recursive formula

$$T_0(x)=1,\quad T_1(x)=2x,$$
$$T_{m+1}(x)=2xT_m(x)-T_{m-1}(x),\quad m=1,2,3,\cdots.$$

Chebyshev wavelet method (CWM)

In the present paper, we consider the fifth order boundary value problems of the form

$$y^{(v)}(x)=g(x)+f(y),\ 0<x<b, \tag{3}$$

with boundary conditions $y(0)=\alpha_0,\ y'(0)=\alpha_1,\ y''(0)=\alpha_2,\ y(b)=\beta_0,\ y'(b)=\beta_1,$ where $g(x)$ is a source term function, $f(y)$ is a given continuous linear or nonlinear function and $\alpha_i,\ i=0,1,2$ and $\beta_i,\ i=0,1$ are real finite constants. The solution of the Equation (3) can be expanded as a Chebyshev wavelets series as follows:

$$y(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{n,m}\psi_{n,m}(x),$$

where $\psi_{n,m}(x)$ is given by Equation (1). We approximate $y(x)$ by the truncated series

$$y_{k,M}(x)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{n,m}(x). \tag{4}$$

Then a total number of $2^{k-1}M$ conditions should exist for determination of $2^{k-1}M$ coefficients

$$c_{10,}c_{11},\ldots,c_{1M-1},\ c_{20,}c_{21},\ldots,c_{2M-1},\ldots,c_{2^{k-1}0,}c_{2^{k-1}1},\cdots,c_{2^{k-1}M-1}.$$

Since five conditions are furnished by the boundary conditions, namely

$$\begin{aligned}
&u_{k,M}(0)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{n,m}(0)=\alpha_0,\\
&\frac{d}{dx}u_{k,M}(0)=\frac{d}{dx}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{n,m}(0)=\alpha_1,\\
&\frac{d^2}{dx^2}u_{k,M}(0)=\frac{d^2}{dx^2}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{n,m}(0)=\alpha_2,\\
&u_{k,M}(b)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{n,m}(b)=\beta_0,\\
&\frac{d}{dx}u_{k,M}(b)=\frac{d}{dx}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{n,m}(b)=\beta_1,
\end{aligned} \tag{5}$$

We see that there should be $2^{k-1}M-5$ extra conditions to recover the unknown coefficients c_{nm}. These conditions can be obtained by substituting Equation (4) in Equation (3);

$$\frac{d^5}{dx^5}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-5}c_{nm}\psi_{n,m}(x)=f(x)+\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-5}c_{nm}\psi_{n,m}(x). \tag{6}$$

Table 1. Numerical results of example 1.

x	Exact solution	Approximate solution	Error in CWM
0.0	0.000000000000000	0.000000000000000	1.91621E-1001
0.1	0.099465382626808	0.099465382626808	5.04606E-69
0.2	0.195424441305627	0.195424441305627	3.58487E-68
0.3	0.283470349590961	0.283470349590961	1.05735E-67
0.4	0.358037927433905	0.358037927433905	2.14472E-67
0.5	0.412180317675032	0.412180317675032	3.48270E-67
0.6	0.437308512093722	0.437308512093722	4.79781E-67
0.7	0.422888068568800	0.422888068568800	5.68103E-67
0.8	0.356086548558795	0.356086548558795	5.58785E-67
0.9	0.221364280004125	0.221364280004125	3.83830E-67
1.0	0.000000000000000	-0.000000000000000	3.26144E-1001

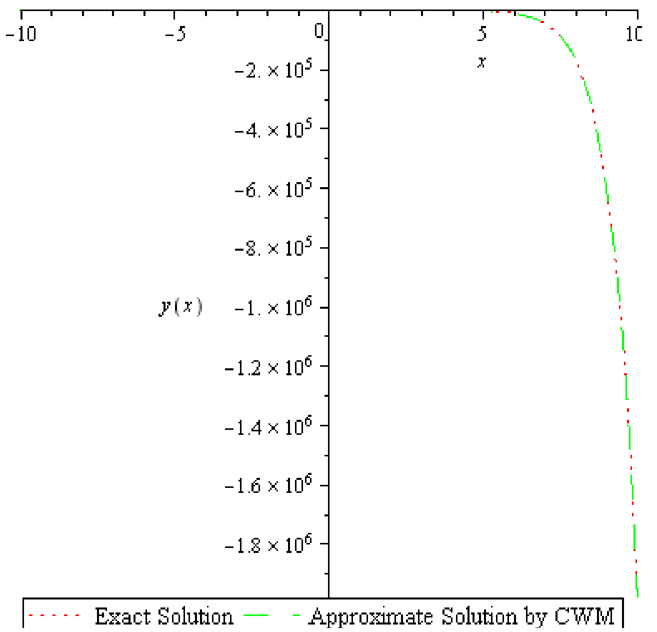

Figure 1. The plot of exact solution and solution obtained by CWM when M=50, k=1 for Example 1.

We, now assume Equation (6) is exact at $2^{k-1}M-5$ points x_i as follows:

$$\frac{d^5}{dx^5}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-5}c_{nm}\psi_{n,m}(x_i)=f(x_i)+\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-5}c_{nm}\psi_{n,m}(x_i). \quad (7)$$

The best choice of the x_i points are the zeros of the shifted Chebyshev polynomials of degree $2^{k-1}M-5$ in the interval $[0,1]$ that is $x_i=\frac{s_i+1}{2}$, where

$$s_i=\cos\left(\frac{(2i-1)\pi}{2^{k-1}M-1}\right),\ i=1,\dots,2^{k-1}M-5.$$

Combine Equations (5) and (7) to obtain $2^{k-1}M$ linear equations from which we can compute values for the unknown coefficients, c_{nm}. Same procedure is repeated for sixth order boundary value problems also.

SOLUTION PROCEDURE

Example 1

Consider the following linear boundary value problem

$$y^{(v)}(x)=y-15e^x-10xe^x,\quad o<x<1,$$

subject to the boundary conditions

$$y(0)=0,\quad y'(0)=1,\quad y''(0)=0,\quad y(1)=0,\quad y'(1)=-e.$$

The theoretical solution for this problem is $y(x)=x(1-x)e^x$. Table 1 shows the comparison of the absolute error between exact solution and approximate solution for M=50 and k=1 by CWM (Figure 1).

Example 2

Consider the following nonlinear boundary value problem

$$y^{(v)}(x)=e^{-x}y^2(x),\quad o<x<1,$$

Subject to the boundary conditions

$$y(0)=1,\quad y'(0)=1,\quad y''(0)=1,\quad y(1)=e,\quad y'(1)=e.$$

The theoretical solution for this problem is $y(x)=e^x$. Table 2 shows the comparison of the absolute error between exact solution and approximate solution for

Table 2. Numerical results of Example 2.

x	Exact solution	Approximate solution	Error in CWM
0.0	1.000000000000000	1.000000000000000	2.00000E-1000
0.1	0.994653826268083	0.994653826268047	3.53979E-14
0.2	0.977122206528136	0.977122206528068	6.77387E-14
0.3	0.944901165303202	0.944901165303108	9.41554E-14
0.4	0.895094818584762	0.895094818584650	1.12161E-13
0.5	0.824360635350064	0.824360635349944	1.19840E-13
0.6	0.728847520156204	0.728847520156088	1.16035E-13
0.7	0.604125812241143	0.604125812241042	1.00544E-13
0.8	0.445108185698494	0.445108185698419	7.43001E-14
0.9	0.245960311115695	0.245960311115655	3.95604E-14
1.0	0.000000000000000	-0.000000000000000	4.20000E-999

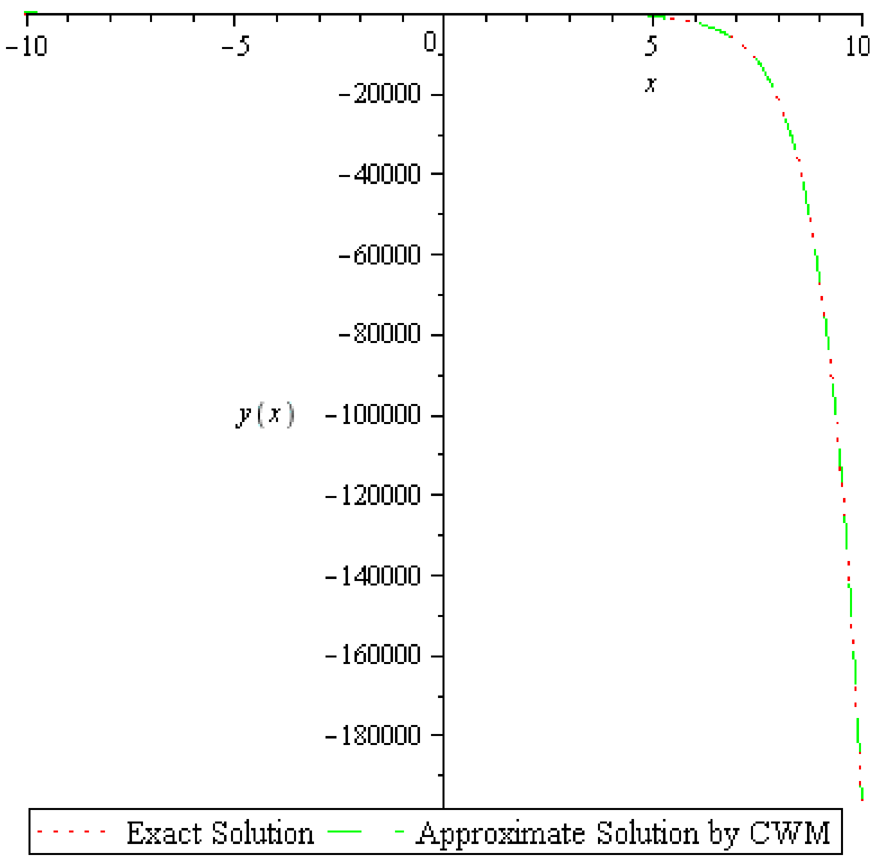

Figure 2. The plot of exact solution and solution obtained by CWM when M=20, k=1 for Example 2.

M=20 and k=1 by CWM (Figure 2).

Example 3

Consider the following linear boundary value problem

$$y^{(vi)}(x) = -6e^x + y(x), \quad 0 < x < 1,$$

Subject to the boundary conditions

$$y(0)=1, \quad y''(0)=-1, \quad y^{(iv)}(0)=-3, \quad y(1)=0, \quad y''(1)=-2e, \quad y^{(iv)}(0)=-4e.$$

The theoretical solution for this problem is $y(x)=(1-x)e^x$. Table 3 shows the comparison of the

Table 3. Numerical results of Example 3.

x	Exact solution	Approximate solution	Error in CWM
0.0	1.000000000000000	1.000000000000000	0.00000E+00
0.1	1.105170918075648	1.105170918075648	2.66320E-20
0.2	1.221402758160170	1.221402758160170	1.88103E-19
0.3	1.349858807576003	1.349858807576003	5.50637E-19
0.4	1.491824697641270	1.491824697641270	1.10563E-18
0.5	1.648721270700128	1.648721270700128	1.76961E-18
0.6	1.822118800390509	1.822118800390509	2.38429E-18
0.7	2.013752707470477	2.013752707470477	2.71668E-18
0.8	2.225540928492468	2.225540928492468	2.45949E-18
0.9	2.459603111156950	2.459603111156950	1.31212E-18
1.0	2.718281828459045	2.718281828459045	2.00000E-999

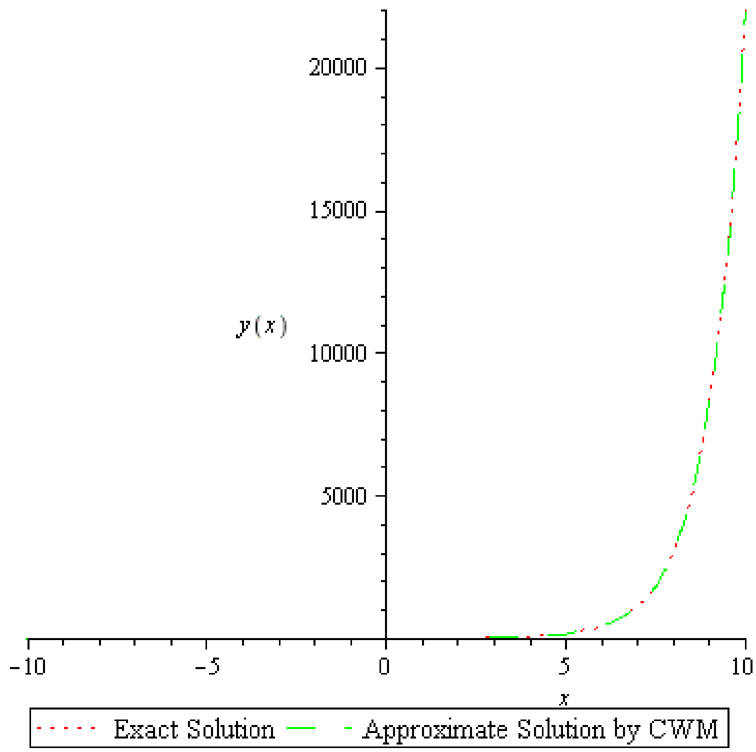

Figure 3. The plot of exact solution and solution obtained by CWM when M=20, k=1 for Example 3.

absolute error between exact solution and approximate solution for M=20 and k=1 by CWM (Figure 3).

Example 4

Consider the following nonlinear boundary value problem

$y^{(vi)}(x) = e^{-x} y^2(x), \quad o < x < 1,$ Subject to the boundary conditions

$$y(0)=1, \quad y''(0)=1, \quad y^{(iv)}(0)=1, \quad y(1)=e, \quad y''(1)=e, \quad y^{(iv)}(1)=e.$$

The theoretical solution for this problem is $y(x)=e^x$. Table 4 shows the comparison of the absolute error between exact solution and approximate solution for M=20 and k=1 by CWM (Figure 4).

CONCULSION

Linear and nonlinear boundary value problems of fifth

Table 4. Numerical results of Example 4.

x	Exact solution	Approximate solution	Error in CWM
0.0	1.000000000000000	1.000000000000000	1.00000E-999
0.1	1.105170918075648	1.105170918075650	1.88721E-15
0.2	1.221402758160170	1.221402758160173	3.61146E-15
0.3	1.349858807576003	1.349858807576008	5.01991E-15
0.4	1.491824697641270	1.491824697641276	5.97998E-15
0.5	1.648721270700128	1.648721270700135	6.38947E-15
0.6	1.822118800390509	1.822118800390515	6.18674E-15
0.7	2.013752707470477	2.013752707470482	5.36084E-15
0.8	2.225540928492468	2.225540928492472	3.96164E-15
0.9	2.459603111156950	2.459603111156952	2.10936E-15
1.0	2.718281828459045	2.718281828459045	1.00000E-999

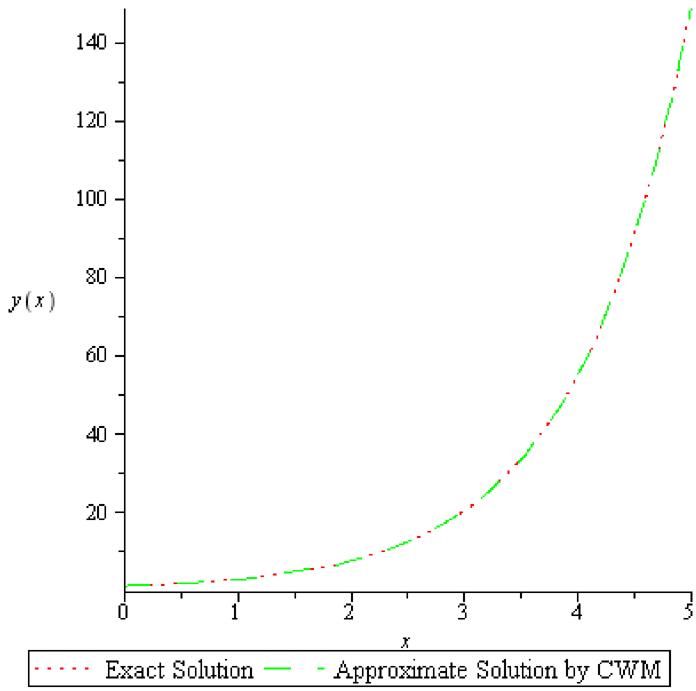

Figure 4. The plot of exact solution and solution obtained by CWM when M=20, k=1 for Example 4.

and sixth order are successfully handled by using CWM. Computational work and numerical results explicitly reflect that CWM is very user-friendly but extremely accurate. It is also concluded that the same CWM may be extended to other linear and nonlinear diversified physical problems of complex nature.

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