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Full Length Research Paper

The dual exponential mapping on dual rotations

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Dual numbers are introduced by Clifford in the mid-nineteenth century and they are systematically applied to kinematics by Study (1903) and Kotelnikov (1895). Any point on the dual unit sphere (DUS) corresponds to a straight line in $\mathbb{R}^3$ (real three space), and vice versa. By this way, there is a one to one correspondence between the dual curves on DUS and the one parameter rigid body motions in $\mathbb{R}^3$. Using Cayley Mapping (McCarthy, 1990) we get a relation between the dual rotation matrices and the dual skew symmetric matrices. In this paper, this relation is given by the exponential mapping which can be called the dual exponential mapping.

Keywords: Kinematics, study mapping, cayley mapping, dual exponential mapping.

INTRODUCTION

Study mapping states relation between the elements of the line space in $\mathbb{R}^3$ and the points of dual unit sphere (DUS). By this method, the rigid body motion in $\mathbb{R}^3$ is represented by a path on DUS. This path can be constructed by the rotations of DUS, hence the rotation (orthogonal) matrices are very important for this construction. Using the Cayley mapping, we find skew symmetric matrices from the rotation matrices. These skew symmetric matrices define the dual rotation angles and the dual Rodrigues vectors (Karakılıç, 2010; Gürsoy and Karakılıç, 2011) which charecterizes the rigid body motion in $\mathbb{R}^3$.

The exponential mapping is a different way of finding skew symmetric matrices (the elements of the algebra $so(3)$) from the rotation matrices (the elements of the group $SO(3)$).

In this paper, a brief explanation will be given for the Cayley mapping on the rotations of DUS and then the practical method, the dual exponential mapping will be introduced as an alternative way of finding dual skew symmetric matrices from the dual rotation matrices.

Preliminaries

A dual number is a formal sum $\hat{a} = a + \varepsilon\, a^*$, where a and a^* are real numbers and $\varepsilon^2 = 0$, similar to the complex unit $i^2 = -1$. The following are addition and multiplication rules for dual numbers;

$$\hat{a} + \hat{b} = (a_1 + \varepsilon a_1^*) + (a_2 + \varepsilon a_2^*) = (a_1 + a_2) + \varepsilon(a_1^* + a_2^*)$$

$$\hat{a}.\hat{b} = (a_1 + \varepsilon a_1^*).(a_2 + \varepsilon a_2^*) = (a_1.a_2) + \varepsilon(a_1 a_2^* + a_2 a_1^*)$$

(In this paper, hat over an alphabet is used for the dual cases; dual number, dual vector, dual matrix, etc.)

If f is a real analytic function, that is, a function

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Abbreviation: DUS, Dual unit sphere.

represented by a power series, $f(x)=\sum_{k=0}^{\infty} a_k\,(x-x_0)^k$, which converges in some interval. We can extend its definition to the ring of dual numbers by letting;

$$f(x+\varepsilon x^*)=\sum_{k=0}^{\infty} a_k(x+\varepsilon x^*-x_0)^k=\sum_{k=0}^{\infty} a_k(x-x_0)^k+\varepsilon\sum_{k=0}^{\infty} k\,a_k(x-x_0)^{k-1}x^* = f(x)+\varepsilon x^* f'(x)$$

For instance,

$$\sin\hat{x}=\sin(x+\varepsilon x^*)=\sin x+\varepsilon x^*\cos x,$$

$$\cos\hat{x}=\cos(x+\varepsilon x^*)=\cos x-\varepsilon x^*\sin x,$$

$$\tan\hat{x}=\tan(x+\varepsilon x^*)=\tan x+\varepsilon x^*(1+\tan^2 x),$$

$$e^{\hat{x}}=e^{(x+\varepsilon x^*)}=e^x e^{\varepsilon x^*}=e^x(1+\varepsilon x^*)=e^x+\varepsilon x^* e^x,$$

$$(\hat{x})^k=(x+\varepsilon x^*)^k=x^k+\varepsilon k x^{k-1}x^*.$$

A dual vector $\vec{\hat{v}}$ in three dimensional dual space ID^3 can be defined by $\vec{\hat{v}}=\vec{v}+\varepsilon\,\overrightarrow{v^*}$, where $\vec{v},\ \overrightarrow{v^*}\in\mathbb{R}^3$. ID^3 is a linear space over the real numbers with dimensions 6. This bilinear form defines a kind of degenerate scalar product which induces a "norm" denoted by $\|.\|$. Hence,

$$\|\vec{\hat{v}}\|=(\vec{\hat{v}}.\vec{\hat{v}})^{1/2}=\left[(\vec{v}+\varepsilon\overrightarrow{v^*})(\vec{v}+\varepsilon\overrightarrow{v^*})\right]^{\frac{1}{2}}$$

$$=\left[\vec{v}\vec{v}+2\varepsilon\vec{v}\overrightarrow{v^*}\right]^{\frac{1}{2}}=\|\vec{v}\|\left(1+2\varepsilon\frac{\vec{v}\overrightarrow{v^*}}{\|\vec{v}\|^2}\right)^{\frac{1}{2}}$$

$$=\|\vec{v}\|\left[\left(1+\varepsilon\frac{\vec{v}\overrightarrow{v^*}}{\|\vec{v}\|^2}\right)^2\right]^{\frac{1}{2}}=\|\vec{v}\|\left(1+\varepsilon\frac{\vec{v}\overrightarrow{v^*}}{\|\vec{v}\|^2}\right)$$

$$=\|\vec{v}\|+\varepsilon\frac{\vec{v}\overrightarrow{v^*}}{\|\vec{v}\|^2}=\left(\|\vec{v}\|,\ \frac{\vec{v}\overrightarrow{v^*}}{\|\vec{v}\|^2}\right) \qquad 1$$

By (1) a dual unit vector $\vec{\hat{v}}=\vec{v}+\varepsilon\overrightarrow{v^*}$ has $\|\vec{v}\|=1$ and $\vec{v}\overrightarrow{v^*}=0$. The set of all dual unit vectors $\left\{\vec{\hat{v}}=\vec{v}+\varepsilon\overrightarrow{v^*}\ \middle|\ \|\vec{\hat{v}}\|=(1,0);\ \vec{v},\overrightarrow{v^*}\in\mathbb{R}^3\right\}$ define the DUS which is also called the Study sphere. An oriented line l is uniquely defined by a point $\vec{p}\in l$ and a unit direction vector $\vec{g}$. The moment vector $\overrightarrow{g^*}$ is defined by $\overrightarrow{g^*}=\vec{p}\times\vec{g}$, which is the smallest distance from l to the origin. Since $\vec{g}.\vec{g}=1$ and $\vec{g}.\overrightarrow{g^*}=0$, the dual vector $\vec{\hat{g}}=\vec{g}+\varepsilon\overrightarrow{g^*}=(\vec{g},\ \overrightarrow{g^*})$ corresponds to a point on DUS.

Study mapping

The mapping which assigns to an oriented line of the Euclidean space a dual vector on DUS is called the Study mapping. If $\vec{\hat{\phi}}=\phi+\varepsilon\phi^*$ is the angle between the dual vectors $\vec{\hat{g}}$ and $\vec{\hat{h}}$ on DUS, then ϕ is the angle and ϕ^* is the smallest distance between the corresponding lines G and H in $\mathbb{R}^3$ (Pottmann and Wallner, 2001).

THE DUAL CAYLEY MAPPING

We can transform a point $\hat{x}$ to a point $\hat{X}$ by a rotation matrix $\hat{A}$ on DUS. Hence $\hat{X}=\hat{A}\hat{x}$. Because of the rigidity of this transformation we have $\|\hat{X}\|=\|\hat{x}\|$ that is,

$$\|\hat{X}\|^2=\hat{X}^T\hat{X}=(\hat{A}\hat{x})^T\hat{A}\hat{x}=\hat{x}^T\hat{A}^T\hat{A}\hat{x}=\hat{x}^T\hat{x}=\|\hat{x}\|^2 \qquad 2$$

Superscript T is used for the transpose of a matrix. Equation 2 shows the orthogonality of $\hat{A}$.

On the other hand, equality of norms $\|\hat{X}\|=\sqrt{\hat{X}^T\hat{X}}=\sqrt{\hat{x}^T\hat{A}^T\hat{A}\hat{x}}=\sqrt{\hat{x}^T\hat{x}}=\|\hat{x}\|$ implies $\hat{X}^T\hat{X}=\hat{x}^T\hat{x}$ and then we have

$$(\hat{X}-\hat{x})^T(\hat{X}+\hat{x})=\hat{X}^T\hat{X}+\hat{X}^T\hat{x}-\hat{x}^T\hat{X}-\hat{x}^T\hat{x}=\hat{X}^T\hat{x}-\hat{x}^T\hat{X}=0$$

This expresses the orthogonality of $(\hat{X}-\hat{x})$ and $(\hat{X}+\hat{x})$. Since $\hat{X}=\hat{A}\hat{x}$,

$\hat{X}+\hat{x}=(\hat{A}+I)\hat{x}$ or $\hat{x}=(\hat{A}+I)^{-1}(\hat{X}+\hat{x})$ and $(\hat{X}-\hat{x})=(\hat{A}-I)\hat{x}$.

Therefore;

$$\hat{X} - \hat{x} = (\hat{A} - I)\left(\hat{A} + I\right)^{-1}(\hat{X} + \hat{x}).$$

Let us denote $(\hat{A} - I)\left(\hat{A} + I\right)^{-1}$ by $\hat{B}$. Since $\hat{X} - \hat{x}$ is orthogonal to $\hat{X} + \hat{x}$. $\hat{B}(\hat{X} + \hat{x})$ is orthogonal to $\hat{X} + \hat{x}$. For a general dual vector $\vec{\hat{v}}$, $\hat{B}\vec{\hat{v}}$ is orthogonal to $\vec{\hat{v}}$. Then we have

$$\vec{\hat{v}}^T \hat{B} \vec{\hat{v}} = \sum(\hat{b}_{ij} + \hat{b}_{ji})\hat{v}_i \hat{v}_j = 0,$$

For all $\vec{\hat{v}}$ on DUS. Hence $\hat{b}_{ii} = 0$ and $\hat{b}_{ij} = -\hat{b}_{ji}$. Which implies the skew symmetry of $\hat{B}$. The skew symmetry of $\hat{B}$ provides $(I - \hat{B})$ not to be singular. Then

$$\hat{B} = \left(\hat{A} - I\right)\left(\hat{A} + I\right)^{-1} \quad \rightarrow \quad \hat{A} = \left(I + \hat{B}\right)(I - \hat{B})^{-1}$$

Hence, we get the Cayley Formula for the dual case:

$$\hat{A} = \left(I + \hat{B}\right)(I - \hat{B})^{-1} \qquad 3$$

Let us compute $\hat{A}^T$;

$$\hat{A}^T = (I + \hat{B})^T((I - \hat{B})^{-1})^T$$

$$= (I + \hat{B}^T)(I - \hat{B}^T)^{-1}.$$

Since $\hat{B}$ is skew symmetric,

$$\hat{A}^T = \left(I - \hat{B}\right)\left(I + \hat{B}\right)^{-1}.$$

In fact

$$\hat{A}\hat{A}^T = \hat{A}^T\hat{A} = I.$$

Hence every skew symmetric dual matrix $\hat{B}$ determines an orthogonal dual matrix $\hat{A}$. If we define the skew symmetric dual matrix $\hat{B}$ by

$$\hat{B} = \begin{pmatrix} 0 & -\hat{b}_3 & \hat{b}_2 \\ \hat{b}_3 & 0 & -\hat{b}_1 \\ -\hat{b}_2 & \hat{b}_1 & 0 \end{pmatrix}$$

then instead of $\hat{B}\vec{\hat{v}}$ one can use $\vec{\hat{b}} \times \vec{\hat{v}}$ where $\vec{\hat{b}} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$. Hence

$$\hat{B}\vec{\hat{v}} = \vec{\hat{b}} \times \vec{\hat{v}}$$

Dual Rodrigues' equations

Given a dual orthogonal matrix $\hat{A}$, a dual skew symmetric matrix $\hat{B}$ is obtained by the Cayley's Formula. It is clear that the relation.

$$\hat{X} - \hat{x} = \hat{B}(\hat{X} + \hat{x})$$

can be written in the form

$$\hat{X} - \hat{x} = \vec{\hat{b}} \times (\hat{X} + \hat{x})$$

This is analogous to the Rodrigues' equations in the real case. $\vec{\hat{b}}$ is called the dual Rodrigues' vector.

If $\hat{\phi}$ is the rotation angle defined for $\hat{A}$, then

$$\left\|\vec{\hat{b}}\right\| = \tan\frac{\hat{\phi}}{2} \qquad 4$$

using the properties of dual numbers (4) implies

$$\|b\| + \varepsilon\frac{b\,b^*}{\|b\|} = \tan\frac{\phi}{2} + \varepsilon\frac{\phi^*}{2}\left(1 + \tan^2\frac{\phi}{2}\right) \qquad 5$$

If we compute a dual unit vector $\vec{\hat{s}}$ from $\vec{\hat{b}}$, that is $\vec{\hat{s}} = \frac{\vec{\hat{b}}}{\|\vec{\hat{b}}\|}$, then the dual skew symmetric matrix $\hat{B}$ can be written as , $\hat{B} = \left\|\vec{\hat{b}}\right\|\hat{S}$, where

$$\hat{S} = \begin{pmatrix} 0 & -\hat{s}_3 & \hat{s}_2 \\ \hat{s}_3 & 0 & -\hat{s}_1 \\ -\hat{s}_2 & \hat{s}_1 & 0 \end{pmatrix}$$

It is clear from (4) that $\hat{B} = \tan\frac{\hat{\phi}}{2}\hat{S}$. Combining this result and the Cayley Formula (3) we get

$$\hat{A} = \left[\cos\left(\tfrac{\hat{\phi}}{2}\right)I - \sin\left(\tfrac{\hat{\phi}}{2}\right)\hat{S}\right]^{-1}\left[\cos\left(\tfrac{\hat{\phi}}{2}\right)I + \sin\left(\tfrac{\hat{\phi}}{2}\right)\hat{S}\right].$$

The inverse $\left[\cos\left(\frac{\hat{\phi}}{2}\right) I - \sin\left(\frac{\hat{\phi}}{2}\right)\hat{S}\right]^{-1}$ can be expanded as,

$$\left[\cos\left(\frac{\hat{\phi}}{2}\right) I - \sin\left(\frac{\hat{\phi}}{2}\right)\hat{S}\right]^{-1} = \cos\left(\frac{\hat{\phi}}{2}\right) I + \sin\left(\frac{\hat{\phi}}{2}\right)\hat{S} + \frac{\sin^2\left(\frac{\hat{\phi}}{2}\right)}{\cos\left(\frac{\hat{\phi}}{2}\right)}\left[I + \hat{S}^2\right] \quad 6$$

Therefore multiplying Equation 6 with $\left[\cos\left(\frac{\hat{\phi}}{2}\right) I + \sin\left(\frac{\hat{\phi}}{2}\right)\hat{S}\right]$ we obtain

$$\hat{A} = I + 2\sin\left(\frac{\hat{\phi}}{2}\right)\cos\left(\frac{\hat{\phi}}{2}\right)\hat{S} + 2\sin^2\frac{\hat{\phi}}{2}\hat{S}^2.$$

This equation uses the identity $\hat{S}^3 + \hat{S} = 0$ which is obtained from the fact that a matrix satisfies its own characteristic equation. The trigonometric half-angle identities further simplify this to the form:

$$\hat{A} = I + \sin\hat{\phi}\,\hat{S} + (1 - \cos\hat{\phi})\hat{S}^2 \quad 7$$

THE DUAL MATRIX EXPONENTIAL

The exponential mapping is an alternative method for finding a relation between the rotation matrices and the skew symmetric matrices (Mampetta, 2006). The dual exponential mapping from $so(3) \times \mathbb{D}$ (the set of dual 3×3 skew symmetric matrices) to $SO(3) \times \mathbb{D}$ (the set of dual 3×3 orthogonal or rotation matrices) (Park, 1994) and (Selig, 2004) obtains the skew symmetric matrix $\hat{B}$ from the orthogonal matrix $\hat{A}$ as in the case of Cayley mapping. The direct calculation shows that a 3×3 dual skew symmetric matrix;

$$\hat{B} = \begin{pmatrix} 0 & -\hat{b}_3 & \hat{b}_2 \\ \hat{b}_3 & 0 & -\hat{b}_1 \\ -\hat{b}_2 & \hat{b}_1 & 0 \end{pmatrix}$$

Satisfies a cubic equation (see Gallier and Xu, 2002 for the real case) obtained from its characteristic equation, that is

$$\hat{B}^3 + \hat{\theta}^2\hat{B} = 0,$$

where $\hat{\theta}^2 = \hat{b}_1^2 + \hat{b}_2^2 + \hat{b}_3^2$, $\hat{B} = B + \varepsilon B^*$, $\hat{\theta} = \theta + \varepsilon\theta^*$, $\hat{b}_i = b_i + \varepsilon b_i^*$, $i = 1,2,3$. A systematic approach will be developed to find the exponential in $so(3) \times \mathbb{D}$. This involves writing the skew symmetric matrix as a sum of mutually annihilating idempotents (see Selig, 2005 for the real case). For the real case this was done by expanding the reciprocal of the cubic into partial fractions, see Curtis (1979).

Let us consider three matrices:

$$\hat{P}_0 = \frac{1}{\hat{\theta}^2}(\hat{B} - i\hat{\theta}I)(\hat{B} + i\hat{\theta}I)$$

$$= \frac{1}{\hat{\theta}^2}\left(\hat{B}^2 + \hat{\theta}^2 I\right) = \frac{1}{\hat{\theta}^2}\hat{B}^2 + I,$$

$$\hat{P}_+ = \frac{-1}{2\hat{\theta}^2}\hat{B}(\hat{B} - i\hat{\theta}I) = \frac{-1}{2\hat{\theta}^2}\hat{B}^2 + \frac{i}{2\hat{\theta}}\hat{B},$$

$$\hat{P}_- = \frac{-1}{2\hat{\theta}^2}\hat{B}(\hat{B} + i\hat{\theta}I) = \frac{-1}{2\hat{\theta}^2}\hat{B}^2 - \frac{i}{2\hat{\theta}}\hat{B},$$

which annihilate each other, that is $\hat{P}_0\hat{P}_+ = \hat{P}_0\hat{P}_- = \hat{P}_+\hat{P}_- = 0$. These dual annihilating matrices can be found by expanding the reciprocal of the cubic into partial fractions (Sobczyk, 1997). Also, it is easy to see that the sum of the dual matrices is the identity matrix

$$\hat{P}_0 + \hat{P}_+ + \hat{P}_- = I.$$

The fact that these dual annihilating matrices are idempotents is now easily proved, for instance,

$$\hat{P}_0 = I\hat{P}_0 = (\hat{P}_0 + \hat{P}_+ + \hat{P}_-)\hat{P}_0 = \hat{P}_0^{\,2} + \hat{P}_+\hat{P}_0 + \hat{P}_-\hat{P}_0 = \hat{P}_0^{\,2}$$

and also, $\hat{P}_0^{\,2} = \hat{P}_0$, $\hat{P}_+^{\,2} = \hat{P}_+$, $\hat{P}_-^{\,2} = \hat{P}_-$.

The final property we need is that a linear combination of the idempotents gives $\hat{B}$,

$$i\hat{\theta}\hat{P}_- - i\hat{\theta}\hat{P}_+ = i\hat{\theta}\left(\frac{-1}{2\hat{\theta}^2}\hat{B}^2 - \frac{i}{2\hat{\theta}}\hat{B}\right) - i\hat{\theta}\left(\frac{-1}{2\hat{\theta}^2}\hat{B}^2 + \frac{i}{2\hat{\theta}}\hat{B}\right)$$

$$= \frac{-i}{2\hat{\theta}}\hat{B}^2 + \frac{1}{2}\hat{B} + \frac{i}{2\hat{\theta}}\hat{B}^2 + \frac{1}{2}\hat{B} = \hat{B}.$$

The point of these manipulations is that, if we arise $\hat{B}$ to some power then because the '$\hat{P}$' matrices are mutually annihilating there are no cross terms. Moreover, since the '$\hat{P}$'s are idempotents only their coefficients are effected by the power,

$$\hat{B}^n = (i\hat{\theta}\hat{P}_- - i\hat{\theta}\hat{P}_+)^n = (-i\hat{\theta})^n\hat{P}_+ + (i\hat{\theta})^n\hat{P}_- .$$

Hence the exponential of the dual matrix $\hat{B}$ can be found as

$$e^{\hat{B}} = \hat{P}_0 + e^{-i\hat{\theta}}\hat{P}_+ + e^{i\hat{\theta}}\hat{P}_- .$$

That is,

$$e^{\hat{B}} = I + \hat{B} + \frac{\hat{B}^2}{2!} + \frac{\hat{B}^3}{3!} + \ldots + \frac{\hat{B}^n}{n!} + \cdots$$

$$= I + (i\hat{\theta}\hat{P}_- - i\hat{\theta}\hat{P}_+) + \frac{(i\hat{\theta}\hat{P}_- - i\hat{\theta}\hat{P}_+)^2}{2!} + \cdots + \frac{(i\hat{\theta}\hat{P}_- - i\hat{\theta}\hat{P}_+)^n}{n!} + \cdots$$

$$= I + i\hat{\theta}\hat{P}_- - i\hat{\theta}\hat{P}_+ + \frac{\left((i\hat{\theta})^2\hat{P}_- + (-i\hat{\theta})^2\hat{P}_+\right)}{2!} + \cdots + \frac{\left((i\hat{\theta})^n\hat{P}_- + (-i\hat{\theta})^n\hat{P}_+\right)}{n!}$$

$$= \left(\hat{P}_0 + \hat{P}_+ + \hat{P}_-\right) + i\hat{\theta}\hat{P}_- - i\hat{\theta}\hat{P}_+ + \frac{(i\hat{\theta})^2}{2!}\hat{P}_- + \frac{(-i\hat{\theta})^2}{2!}\hat{P}_+ + \cdots + \frac{(i\hat{\theta})^n}{n!}\hat{P}_- + \frac{(i\hat{\theta})^n}{n!}\hat{P}_+ + \cdots$$

$$= \hat{P}_0 + \hat{P}_+\left(1 - i\hat{\theta} + \frac{(-i\hat{\theta})^2}{2!} + \cdots\right) + \hat{P}_-\left(1 + i\hat{\theta} + \frac{(i\hat{\theta})^2}{2!} + \cdots\right)$$

$$= \hat{P}_0 + e^{-i\hat{\theta}}\hat{P}_+ + e^{i\hat{\theta}}\hat{P}_- .$$

Now we can replace the idempotents by their definitions in terms of $\hat{B}$ to get

$$e^{\hat{B}} = \frac{1}{\hat{\theta}^2}\hat{B}^2 + I_3 + e^{-i\hat{\theta}}\left(-\frac{1}{2\hat{\theta}^2}\hat{B}^2 + \frac{i}{2\hat{\theta}}\hat{B}\right) + e^{i\hat{\theta}}\left(-\frac{1}{2\hat{\theta}^2}\hat{B}^2 - \frac{i}{2\hat{\theta}}\hat{B}\right)$$

$$= I_3 + \frac{i}{2\hat{\theta}}\left(e^{-i\hat{\theta}} - e^{i\hat{\theta}}\right)\hat{B} - \frac{1}{2\hat{\theta}^2}\left(e^{i\hat{\theta}} + e^{-i\hat{\theta}} - 2\right)\hat{B}^2 .$$

Finally, replacing the complex exponential by trigonometric functions we have

$$e^{\hat{B}} = I_3 + \frac{1}{\hat{\theta}}\sin\hat{\theta}\,\hat{B} + \frac{1}{\hat{\theta}^2}\left(1 - \cos\hat{\theta}\right)\hat{B}^2.$$

Moreover, normalize $\hat{B}$ dividing by $\hat{\theta} = \left(\hat{b}_1^2 + \hat{b}_2^2 + \hat{b}_3^2\right)^{\frac{1}{2}}$ to obtain $\hat{B} = \hat{\theta}\hat{S}$. Thus,

$$e^{\hat{B}} = I_3 + \sin\hat{\theta}\,\hat{S} + \left(1 - \cos\hat{\theta}\right)\hat{S}^2 \qquad 8$$

Comparing (8) with (7) it is easy to say that $e^{\hat{B}}$ corresponds to the dual rotation matrix $\hat{A}$.

On the other hand, for a given orthogonal dual matrix $\hat{A}$, one can obtain the angle $\hat{\theta}$ and the skew-symmetric matrix $\hat{B}$ as follows;

Notice that $Tr(I) = 3$, $Tr(\hat{B}) = 0$ and $Tr(\hat{B}^2) = -2\hat{\theta}^2$, so the trace of $\hat{A}$ gives

$$Tr(\hat{A}) = Tr(I) + \frac{1}{\hat{\theta}}\sin\hat{\theta}\,Tr(\hat{B}) + \frac{1}{\hat{\theta}^2}\left(1 - \cos\hat{\theta}\right)Tr(\hat{B}^2)$$

$$= 3 + \frac{1}{\hat{\theta}}.0 + \frac{1}{\hat{\theta}^2}\left(1 - \cos\hat{\theta}\right)\left(-2\hat{\theta}^2\right)$$

$$= 3 - 2 + 2\cos\hat{\theta}$$

$$= 1 + 2\cos\hat{\theta} .$$

Then we have

$$\hat{\theta} = arc\cos\left(\frac{Tr(\hat{A}) - 1}{2}\right) \qquad 9$$

To find the skew-symmetric matrix $\hat{B}$ we observe that, since the matrix $\hat{B}$ is skew-symmetric, its square $\hat{B}^2$ must be symmetric like I. If we compute $\hat{A} - \hat{A}^T$ we obtain

$$\hat{A} - \hat{A}^T = \left(I + \frac{1}{\hat{\theta}}\sin\hat{\theta}\hat{B} + \frac{1}{\hat{\theta}^2}\left(1 - \cos\hat{\theta}\right)\hat{B}^2\right) - \left(I + \frac{1}{\hat{\theta}}\sin\hat{\theta}\hat{B}^T + \frac{1}{\hat{\theta}^2}\left(1 - \cos\hat{\theta}\right)(\hat{B}^T)^2\right)$$

$$= \frac{1}{\hat{\theta}}\sin\hat{\theta}\left(\hat{B} - \hat{B}^T\right) + \frac{1}{\hat{\theta}^2}\left(1 - \cos\hat{\theta}\right)\left(\hat{B}^2 - \left(\hat{B}^T\right)^2\right)$$

$$= \frac{1}{\hat{\theta}}\sin\hat{\theta}\; 2\,\hat{B}$$

$$= \frac{2}{\hat{\theta}}\sin\hat{\theta}\,\hat{B}$$

since $\hat{B} - \hat{B}^T = 2\hat{B}$ and $\hat{B}^2 - \left(\hat{B}^T\right)^2 = 0$.

Thus we have

$$\hat{A} - \hat{A}^T = \frac{2}{\bar{\hat{\theta}}}\sin\hat{\theta}\,\hat{B}$$

and then

$$\hat{B} = \frac{\hat{\theta}}{2\sin\hat{\theta}}\left(\hat{A} - \hat{A}^T\right) \qquad 10$$

Substituting Equation 9 into 10, gives;

$$\hat{B} = \frac{arc\cos\left(\frac{Tr(\hat{A})-1}{2}\right)}{2\sin\left(arc\cos\frac{Tr(\hat{A})-1}{2}\right)}\left(\hat{A} - \hat{A}^T\right).$$

CONCLUSION

The Cayley mapping is the classical way of finding for each dual rotation matrix $\hat{A}$ a dual skew symmetric matrix $\hat{B}$ on DUS, and vice versa. Here the dual skew symmetric matrix $\hat{B}$ characterizes the rotation by defining the dual Rodrigues vector $\hat{b}$ and the dual rotation angle $\hat{\theta}$.

The exponential mapping discussed here is an efficient way of finding dual skew symmetric matrices from the dual orthogonal matrices. If the the dual rotation matrix $\hat{A}$ is given, we can simply compute the dual rotation angle $\hat{\theta}$ and the dual skew symmetric matrices $\hat{B}$, by use of the trace of $\hat{A}$.

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